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Ricci and conformal Ricci solitons on trans-Sasakian space forms with semi-symmetric metric connection

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The aim of this paper is to study the Ricci solitons and conformal Ricci solitons in trans-Sasakian space form with semi-symmetric metric connection.

Key words: Semi-symmetric; trans-Sasakian space form; Ricci solitons; Conformal Ricci solitons

1. Introduction

In 1982 Hamilton [8] introduced the concept of Ricci flow and proved its existence. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1}$$

on a compact Riemannian manifold M with Riemannian metric g, where S is the Ricci tensor. A self-similar solution to the Ricci flow (1) is called a **Ricci soliton** which moves under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow in space of metrics on M. A Ricci soliton is a generalization of an Einstein metric. The Ricci soliton equation is given by

$$\mathcal{L}_X g + 2S = 2\lambda g \tag{2}$$

where $\mathcal{L}$ is the Lie derivative, S is the Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero and negetive respectively.

Fischer during 2003–2004 developed the concept of conformal Ricci flow [6] which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by [7]

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \tag{3}$$

where $R(g) = -1$ and p is a non-dynamical scalar field(time dependent scalar field), $R(g)$ is the scalar curvature of the n-dimessional manifold M.

In 2015, N. Basu and A. Bhattacharyya [1] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g \tag{4}$$

where λ is a scalar.

Several authors [15, 9, 12, 13] have studied Ricci solitons on different types of trans-Sasakian manifolds. Conformal Ricci solitons on trans-Sasakian manifolds are also studied by various authors [4, 10, 2]. But they have studied on trans-Sasakian manifold with Levi-Civita connection. In this article, we have studied Ricci solitons and conformal Ricci solitons on trans-Sasakian manifold with semi-symmetric metric connection and on trans-Sasakian space form with semi-symmetric metric connection.

2. Preliminaries

Definition 2.1. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$ dimensional contact metric manifold, where φ is a (1, 1)-tensor field, ξ a unit vector field and η a smooth 1-form dual to ξ with respect to the Riemannian metric g satisfying

$$\left.\begin{aligned} &\varphi^2 = -I + \eta \otimes \xi, \\ &\varphi(\xi) = 0, \\ &\eta \circ \varphi = 0, \\ &g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \end{aligned}\right\} \tag{5}$$

$X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. If there are smooth functions α, β on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ satisfying

$$\begin{aligned}(\nabla\varphi)(X,Y) = \alpha\,[g(X,Y)\xi - \eta(Y)X] \\ + \beta\,[g(\varphi X, Y)\xi - \eta(Y)\varphi X],\end{aligned} \tag{6}$$

having the property

$$(\nabla\varphi)(X,Y) = \nabla_X \varphi Y - \varphi(\nabla_X Y), \ X, Y \in \mathfrak{X}(M),$$

∇ is the Levi-Civita connection with respect to the metric g. Then the manifold is said to be trans-Sasakian manifold of type (α, β) and denoted by

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$(M, \varphi, \xi, \eta, g, \alpha, \beta)$ [11]. From equations (5) and (6), it follows that

$$\nabla_X \xi = -\alpha\varphi(X) + \beta(X - \eta(X)\xi), X \in \mathfrak{X}(M). \quad (7)$$

The following relations hold in a trans-Sasakian manifold

$$\nabla_X \xi = -\alpha\varphi X + \beta[X - \eta(X)\xi], \quad (8)$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y), \quad (9)$$

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] \\ &+ 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X \\ &- (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y, \quad (10) \end{aligned}$$

$$\begin{aligned} R(\xi,Y)X &= (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(X)Y] \\ &+ 2\alpha\beta[g(\varphi X, Y)\xi - \eta(X)\varphi Y] + (X\alpha)\varphi Y \\ &+ g(\varphi X, Y)(\text{grad } \alpha) + X\beta[Y - \eta(Y)\xi] \\ &- g(\varphi X, \varphi Y)(\text{grad } \beta), \quad (11) \end{aligned}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X], \quad (12)$$

$$\begin{aligned} S(X,\xi) &= [2n(\alpha^2 - \beta^2) - \xi\beta]\eta(X) \\ &\quad - (2n-1)X\beta - (\varphi X)\alpha, \quad (13) \end{aligned}$$

$$\begin{aligned} Q\xi &= [2n(\alpha^2 - \beta^2) - \xi\beta]\xi - (2n-1)\text{grad } \beta \\ &\quad + \varphi(\text{grad } \alpha), \quad (14) \end{aligned}$$

$$S(X,Y) = g(QX,Y) \text{ and } 2\alpha\beta + \xi\alpha = 0. \quad (15)$$

Definition 2.2. A trans-Sasakian manifold M is said to be an η-Einstein manifold [5] if Ricci tensor satisfies the relation

$$S(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y), \quad (16)$$

where λ, μ are smooth functions.

3. Semi-symmetric metric connection and trans-Sasakian space Form

3.1 Semi-symmetric metric connection

Let M be an $m = (2n+1)$-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and ∇ be the Levi-Civita connection on (M^m, g). A linear connection $\widetilde{\nabla}$ defined on (M^m, g) is said to be **semi-symmetric** [3], if its torsion tensor T is of the forms

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] \quad (17)$$

satisfying

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \quad (18)$$

for all $X, Y \in \mathfrak{X}(M)$

where η is an 1-form with associated vector field ξ defined by

$$\eta(X) = g(X, \xi), \quad (19)$$

for all vector fields $X \in \mathfrak{X}(M)$.

A semi-symmetric connection $\widetilde{\nabla}$ is called a **semi-symmetric metric connection** if it further satisfies

$$\widetilde{\nabla} g = 0.$$

A relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ on (M^m, g) has been obtained by Yano [12] which is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \quad (20)$$

Further, a relation between the curvature tensor R of the Levi-Civita connection ∇ and the curvature tensor $\widetilde{R}$ of the semi-symmetric metric connection $\widetilde{\nabla}$ is given by

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X \\ &\quad + g(X,Z)AY - g(Y,Z)AX, \quad (21) \end{aligned}$$

for all vector fields X, Y, Z on M, where α is the (0, 2)-tensor field and A is a tensor field of type (1, 1) defined by

$$\begin{aligned} \alpha(X,Y) &= (\nabla_X \eta)Y - \eta(X)\eta(Y) \\ &\quad + \frac{1}{2}\eta(\xi)g(X,Y), \quad (22) \end{aligned}$$

$$\text{and} \quad \alpha(X,Y) = g(AX, Y). \quad (23)$$

The curvature tensor $\widetilde{R}$ with respect to $\widetilde{\nabla}$ is given by

$$\begin{aligned} \widetilde{R}(X,Y)Z &= \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z \\ &\quad - \widetilde{\nabla}_{[X,Y]} Z. \quad (24) \end{aligned}$$

Using (20), we get

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + \alpha[g(\varphi Y, Z)X \\ &- g(\varphi X, Z)Y + g(Y,Z)\varphi X - g(X,Z)\varphi Y] \\ &+ (2\beta + 1)[g(X,Z)Y - g(Y,Z)X] \\ &- (+1)[\eta(Z)\eta(X)Y - \eta(Z)\eta(Y)X \\ &+ \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi]. \quad (25) \end{aligned}$$

Lemma 3.1. *From equation (25), we have*

$$\widetilde{R}(X,Y)\xi = (\alpha^2-\beta^2-\beta)[\eta(Y)X-\eta(X)Y] + (2\alpha\beta+\alpha)[\eta(Y)\varphi X-\eta(X)\varphi Y] + (Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2X - (X\beta)\varphi^2Y.$$

Lemma 3.2.

$$\widetilde{R}(\xi,Y)\xi = (\alpha^2-\beta^2-\beta-\xi\beta)[\eta(Y)\xi-Y] - (2\alpha\beta+\alpha+\xi\alpha)\varphi Y.$$

Remark 3.3. If α,β are constants, then

$$\widetilde{R}(\xi,Y)\xi = (\alpha^2-\beta^2-\beta)[\eta(Y)\xi-Y] - (2\alpha\beta+\alpha)\varphi Y. \quad (26)$$

The Ricci tensor $\widetilde{S}$ with respect to $\widetilde{\nabla}$ is

$$\widetilde{S}(X,Y) = S(X,Y)+\alpha(2n-1)g(\varphi X,Y) - \{(4n-1)\beta+(2n-1)\}g(X,Y) + (\beta+1)\eta(X)\eta(Y), \quad (27)$$

and scalar curvatur $\widetilde{r}$ is

$$\widetilde{r} = r-8n^2\beta-2n(2n-1). \quad (28)$$

where $S(X,Y)$, r are Ricci tensor and scalar curvature with respect to $\widetilde{\nabla}$ respectively.

Lemma 3.4.

$$\widetilde{S}(X,\xi) = S(X,\xi)+\alpha(2n-1)g(\varphi X,\xi) - \{(4n-1)\beta+(2n-1)\}g(X,\xi) + (\beta+1)\eta(X)\eta(\xi) = S(X,\xi)-2n(2\beta+1)\eta(X)$$

$$\therefore\ \widetilde{S}(X,\xi) = [2n(\alpha^2-\beta^2-2\beta-1)-\xi]\eta(X) - (2n-1)X\beta-(\varphi X)\alpha. \quad (29)$$

Remark 3.5. If α,β are constants, then

$$\widetilde{S}(X,\xi) = 2n(\alpha^2-\beta^2-2\beta-1)\eta(X). \quad (30)$$

Lemma 3.6. $\widetilde{Q}\xi = [2n(\alpha^2-\beta^2-2\beta-1)-\xi\beta]\xi - (2n-1)\text{grad}\,\beta+\varphi(\text{grad}\,\alpha)$.

Remark 3.7. If α,β are constants, then

$$\widetilde{Q}\xi = 2n(\alpha^2-\beta^2-2\beta-1)\xi \quad (31)$$

3.2 Trans-Sasakian space form

A trans-Sasakian manifold M^{2n+1} of constant φ-sectional curvature c is called a **trans-Sasakian space form** [14] denoted by $M^{2n+1}(c)$ and its curvature tensor is given by

$$R(X,Y)Z = \frac{\gamma(c+3)+\delta(c-3)}{4}[g(Y,Z)X-g(X,Z)Y] + \frac{\gamma(c-1)+\delta(c+1)}{4}\{[\eta(X)Y-\eta(Y)X]\eta(Z) + [g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]\xi + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y + 2g(X,\varphi Y)\varphi Z\},$$

where γ and δ are smooth functions.

The Ricci tensor on trans-Sasakian space form defined by

$$S(X,Y) = \frac{1}{2}[c(n+1)(\gamma+\delta)+(3n-1)(\gamma-\delta)] g(X,Y) - \frac{n+1}{2}[c(\gamma+\delta)-(\gamma-\delta)]\eta(X)\eta(Y).$$

By (5), it becomes

$$S(X,Y) = 2ng(X,Y)+\frac{n+1}{2}[c(\gamma+\delta)-(\gamma-\delta)] g(\varphi X,\varphi Y). \quad (32)$$

With the help of (25) and (27), the trans-Sasakian space form with semi-symmetric metric connection is

$$\widetilde{R}(X,Y)Z = \frac{\gamma(c+3)+\delta(c-3)}{4}[g(Y,Z)X-g(X,Z)Y] + \frac{\gamma(c-1)+\delta(c+1)}{4}\{[\eta(X)Y-\eta(Y)X]\eta(Z) + [g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]\xi + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y + 2g(X,\varphi Y)\varphi Z\} + \alpha[g(\varphi Y,Z)X - g(\varphi X,Z)Y + g(Y,Z)\varphi X - g(X,Z)\varphi Y] + (2\beta+1)[g(X,Z)Y-g(Y,Z)X] - (\beta+1)\eta(Z)\eta(X)Y - \eta(Z)\eta(Y)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi]. \quad (33)$$

and the Ricci tensor is

$$\widetilde{S}(X,Y) = \frac{1}{2}[c(n+1)(\gamma+\delta) + (3n-1)(\gamma-\delta)]g(X,Y) - \frac{n+1}{2}[c(\gamma+\delta)-(\gamma-\delta)]\eta(X)\eta(Y) + \alpha(2n-1)g(\varphi X,Y) - \{(4n-1)\beta + (2n-1)\}g(X,Y) + (\beta+1)\eta(X)\eta(Y), \quad (34)$$

Using (5), it can be written as,

$$\widetilde{S}(X,Y) = \frac{n+1}{2}[c(\gamma+\delta)-(\gamma-\delta)]g(\varphi X,\varphi Y)$$

$$+\alpha(2n-1)g(\varphi X,Y)+[1-(4n-1)\beta]g(X,Y) + (\beta+1)\eta(X)\eta(Y). \quad (35)$$

Replacing Y by ξ in (35), we have,

$$\widetilde{S}(X,\xi)=2[1-(2n-1)\beta]\eta(X). \quad (36)$$

4. Ricci Solitons

Let V be pointwise collinear vector field with ξ i.e. $V=b\xi$, where b is a function on the trans-Sasakian manifold. Then $(\mathcal{L}_V g+2\widetilde{S}+2\lambda g)(X,Y)=0$, implies

$$g(\nabla_X b\xi,Y)+g(\nabla_Y b\xi,X)+2\widetilde{S}(X,Y) + 2\lambda g(X,Y)=0,$$

$$\text{or,}\quad bg(-\alpha\varphi X+\beta(X-\eta(X)\xi),Y)+(Xb)\eta(Y) + bg(-\alpha\varphi Y+\beta(Y-\eta(Y)\xi),X) + (Yb)\eta(X)+2\widetilde{S}(X,Y)+2\lambda g(X,Y)=0,$$

which yields

$$2b\beta g(X,Y)-2b\beta\eta(X)\eta(Y)+(Xb)\eta(Y) + (Yb)\eta(X)+2\widetilde{S}(X,Y)+2\lambda g(X,Y)=0. \quad (37)$$

Replacing Y by ξ in (37) it follows that

$$2b\beta\eta(X)-2b\beta\eta(X)+(Xb)+(\xi b)\eta(X) + 2\widetilde{S}(X,\xi)+2\lambda\eta(X)=0.$$

which gives by (29),

$$Xb+\{\xi b+[4n(\alpha^2-\beta^2-2\beta-1)-2\xi\beta] + 2\lambda\}\eta(X)-2(2n-1)X\beta - 2(\varphi X)\alpha=0. \quad (38)$$

Putting $X=\xi$, we have

$$2\xi b+\{[4n(\alpha^2-\beta^2-2\beta-1)-2\xi\beta]+2\lambda\} - 2(2n-1)\xi\beta=0.$$

If α,β are constants, then

$$\xi b=-\{2n(\alpha^2-\beta^2-2\beta-1)+\lambda\}.$$

Hence (38) becomes

$$Xb=-\{2n(\alpha^2-\beta^2-2\beta-1)+\lambda\}\eta(X).$$

$$\text{or,}\quad db=-\{2n(\alpha^2-\beta^2-2\beta-1)+\lambda\}\eta. \quad (39)$$

Applying d on (39), we get $\{2n(\alpha^2-\beta^2-2\beta-1)+\lambda\}d\eta=0$. Since $d\eta\neq 0$ we have

$$2n(\alpha^2-\beta^2-2-1)+\lambda=0. \quad (40)$$

Using (40) in (39) yields b is a constant.
Therefore from (37) it follows

$$\widetilde{S}(X,Y)=-(b\beta+\lambda)g(X,Y)+b\beta\eta(X)\eta(Y).$$

which implies that M is of constant scalar curvature provided α,β are constants. This leads to the following:

Theorem 4.1. *If a trans-Sasakian manifold $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ with semi-symmetric metric connection is a Ricci soliton and V is pointwise collinear vector field with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided α,β are constants.*

Corollary 4.2. *If a trans-Sasakian manifold $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ with semi-symmetric metric connection is a Ricci soliton and V is pointwise collinear vector field with ξ and V is a constant multiple of ξ , then the manifold is $\eta-$Einstein manifold provided α,β are constants.*

The equation $(\mathcal{L}_V g+2\widetilde{S}+2\lambda g)(X,Y)=0$, implies

$$g(\nabla_X b\xi,Y)+g(\nabla_Y b\xi,X)+2\widetilde{S}(X,Y) + 2\lambda g(X,Y)=0,$$

$$\text{or,}\quad bg(-\alpha\varphi X+\beta(X-\eta(X)\xi),Y)+(Xb)\eta(Y) + bg(-\alpha\varphi Y+\beta(Y-\eta(Y)\xi),X)+(Yb)\eta(X) + 2\widetilde{S}(X,Y)+2\lambda g(X,Y)=0,$$

which yields

$$2b\beta g(X,Y)-2b\beta\eta(X)\eta(Y)+(Xb)\eta(Y) + (Yb)\eta(X)+2\widetilde{S}(X,Y) + 2\lambda g(X,Y)=0. \quad (41)$$

Replacing Y by ξ it follows that

$$2b\beta\eta(X)-2b\beta\eta(X)+(Xb)+(\xi b)\eta(X) + 2\widetilde{S}(X,\xi)+2\lambda\eta(X)=0.$$

Using (35),

$$Xb+[4\{1-(2n-1)\beta\}+\xi b+2\lambda]\eta(X)=0. \quad (42)$$

Replacing X by ξ, we have

$$\xi b=-2\{1-(2n-1)\beta\}-\lambda.$$

Hence (42) becomes

$$Xb=-[2\{1-(2n-1)\}+\lambda]\eta(X).$$

$$\text{or,}\quad db=-[2\{1-(2n-1)\beta\}+\lambda]\eta. \quad (43)$$

Applying d on (43), we get $[2\{1-(2n-1)\beta\}+\lambda]d\eta=0$. Since $d\eta\neq 0$ we have

$$2\{1-(2n-1)\beta\}+\lambda=0. \quad (44)$$

Using (44) in (43) yields b is a constant.
Therefore from (37) it follows

$$\widetilde{S}(X,Y)=-(b+\lambda)g(X,Y)+b\eta(X)\eta(Y).$$

Theorem 4.3. *If a tran-Sasakian space form $(M,\varphi,\xi,\eta,g,c,\alpha,\beta)$ with semi-symmetric metric connection is a Ricci soliton and V is pointwise collinear vector field with ξ, then V is a constant multiple of ξ.*

Corollary 4.4. If a tran-Sasakian space form $(M,\varphi,\xi,\eta,g,c,\alpha,\beta)$ with semi-symmetric metric connection is a Ricci soliton and V is pointwise collinear vector field with ξ and V is a constant multiple of ξ, then it is $\eta-$Einstein provided_ is constant.

5. Conformal Ricci solitons

A conformal Ricci soliton equation on a Riemannian manifold M is defined by

$$\mathcal{L}_V g + 2S = \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g, \tag{45}$$

where V is a vector field.

Let V be pointwise colinear with ξ i.e. $V = b\xi$ where b is a function on the trans-Sasakian manifold. Then

$$\left(\mathcal{L}_{b\xi} g + 2S - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g\right)(X,Y) = 0,$$

which implies

$$(\mathcal{L}_{b\xi} g)(X,Y) + 2S(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0. \tag{46}$$

Repalcing S by $\widetilde{S}$ in equation (46), we have

$$(\mathcal{L}_{b\xi} g)(X,Y) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0,$$

which implies

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0,$$

or,
$$bg(-\alpha\varphi X + \beta(X - \eta(X)\xi), Y) + (Xb)\eta(Y) + bg(-\alpha\varphi Y + \beta(Y - \eta(Y)\xi), X) + (Yb)\eta(X) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0,$$

which yields

$$2b\beta g(X,Y) - 2b\beta\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0. \tag{47}$$

Replacing Y by ξ it follows that

$$2b\beta\eta(X) - 2b\beta\eta(X) + (Xb) + (\xi b)\eta(X) + 2\widetilde{S}(X,\xi) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] \eta(X) = 0.$$

Using (29),

$$Xb + \left\{\xi b + [4n(\alpha^2 - \beta^2 - 2\beta - 1) - 2\xi\beta] - 2\lambda + \left(p + \frac{2}{3}\right)\right\}\eta(X) - 2(2n-1)X\beta - 2(\varphi X)\alpha = 0. \tag{48}$$

Put $X = \xi$, we have

$$2\xi b + \left\{[4n(\alpha^2 - \beta^2 - 2\beta - 1) - 2\xi\beta] - 2\lambda + \left(p + \frac{2}{3}\right)\right\} - 2(2n-1)\xi\beta = 0.$$

If α, β are constants, then

$$\xi b = -\left\{2n(\alpha^2 - \beta^2 - 2\beta - 1) - \lambda + \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}.$$

Hence (48) becomes

$$Xb = \left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2n(\alpha^2 - \beta^2 - 2\beta - 1)\right\}\eta(X).$$

or,
$$db = \left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2n(\alpha^2 - \beta^2 - 2\beta - 1)\right\}\eta. \tag{49}$$

Applying d on (49), we get $\left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2n(\alpha^2 - \beta^2 - 2\beta - 1)\right\} d\eta = 0$. Since $d\eta \neq 0$ we have

$$\left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2n(\alpha^2 - \beta^2 - 2\beta - 1)\right\} = 0. \tag{50}$$

Using (50) in (49) yields b is a constant. Therefore from (47) it follows

$$\widetilde{S}(X,Y) = \left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - b\beta\right\} g(X,Y) + b\beta\eta(X)\eta(Y).$$

which implies that M is of constant scalar curvature provided α, β are constants. This leads to the following:

Theorem 5.1. *If a trans-Sasakian manifold $(M,\varphi,\xi,\eta,g,\alpha,\beta)$ with semi-symmetric metric connection is a conformal Ricci soliton and V is pointwise collinear vector field with ξ, then V is a* *constant multiple of ξ and it is of constant scalar curvature provided α, β are constants.*

Corollary 5.2. *If a trans-Sasakian manifold $(M, \varphi, \xi, \eta, g, \alpha, \beta)$ with semi-symmetric metric connection is a Ricci soliton and V is pointwise collinear vector field with ξ and V is a constant multiple of ξ, then it is $\eta-$Einstein manifold provided α, β are constants.*

The $(\mathcal{L}_{b\xi}g)(X,Y) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0$ implies

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0,$$

or,
$$bg(-\alpha\varphi X + \beta(X - \eta(X)\xi), Y) + (Xb)\eta(Y) + bg(-\alpha\varphi Y + \beta(Y - \eta(Y)\xi), X) + (Yb)\eta(X) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0,$$

which yields

$$2b\beta g(X,Y) - 2b\beta\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2\widetilde{S}(X,Y) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,Y) = 0. \quad (51)$$

Replacing Y by ξ it follows that

$$2b\beta\eta(X) - 2b\eta(X) + (Xb) + (\xi b)\eta(X) + 2\widetilde{S}(X,\xi) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] \eta(X) = 0.$$

Using (35),

$$Xb + \left\{\xi b + 4[1 - (2n-1)\beta] - 2\lambda + \left(p + \frac{2}{3}\right)\right\} \eta(X) = 0. \quad (52)$$

Putting $X = \xi$, we have

$$2\xi b + \left\{4[1 - (2n-1)\beta] - 2\lambda + \left(p + \frac{2}{3}\right)\right\} = 0.$$

or,
$$\xi b = -\left\{2[1 - (2n-1)\beta] - \lambda + \frac{1}{2}\left(p + \frac{2}{3}\right)\right\}.$$

Hence (52) becomes

$$Xb = \left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2[1 - (2n-1)\beta]\right\}\eta(X).$$

or,
$$db = \left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2[1 - (2n-1)\beta]\right\}\eta. \quad (53)$$

Applying d on (53), we get $\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2[1 - (2n-1)\beta]\}d\eta = 0$. Since $d\eta \neq 0$ we have

$$\left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - 2[1 - (2n-1)\beta]\right\} = 0. \quad (54)$$

Using (54) in (53) yields b is a constant.

Therefore from (51) it follows

$$\widetilde{S}(X,Y) = \left\{\lambda - \frac{1}{2}\left(p + \frac{2}{3}\right) - b\beta\right\} g(X,Y) + b\beta\eta(X)\eta(Y).$$

which implies that M is of constant scalar curvature provided_ is constants. This leads to the following:

Theorem 5.3. *If a trans-Sasakian space form $(M, \varphi, \xi, \eta, g, c, \alpha, \beta)$ with semi-symmetric metric connection is a confomal Ricci soliton and V is pointwise collinear vector field with ξ, then V is a constant multiple of ξ.*

Corollary 5.4. *If a trans-Sasakian space form $(M, \varphi, \xi, \eta, g, c, \alpha, \beta)$ with semi-symmetric metric connection is a conformal Ricci soliton and V is pointwise collinear vector field with ξ and V is a constant multiple of ξ , then it is $\eta-$Einstein manifold provided_ is constant.*

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