



## Some Properties of K-Yamabe Solitons on Kenmotsu Manifolds

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### Abstract

In this present paper, we have to study k-Yamabe soliton on 3-dimensional Kenmotsu manifold. In the very first, we show that if a 3-dimensional Kenmotsu metric with k-Yamabe soliton is the scalar curvature is constant and the vector field is killing. After that, we prove the manifold is the locally isometric to Hyperbolic space  $H(2n+1)(-1)$  and the manifold becomes Einstein manifold.

**Keywords:** 3-dimensional Kenmotsu manifold, Yamabe solitons, k-Yamabe solitons

### 1. Introduction

The notion of the Yamabe flow introduced by Hamilton [2] at that time when Ricci flow was introduced and used as a process for constructing a metrics of constant scalar curvature in a given conformal class of Riemannian metrics. On a Riemannian manifold  $(Mn, g)$ , Yamabe flow can be precised as the amends of the Riemannian metric by the equation

$$\frac{\delta}{\delta t}g = -rg, \quad g(0) = g_0 \quad (1)$$

where  $r$  is the scalar curvature of  $g$ . One notes that Yamabe flow agry to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. Similar to the Ricci soliton, Yamabe solitons are the self similar solutions of Yamabe flow that operates by a one-parameter family of diffeomorphisms yielded by a fixed (time-independent) vector field  $V$  on  $Mn$ , and homotheties. A Riemannian manifold  $(Mn, g)$  is said to be a Yamabe soliton if there exists a smooth vector field  $V$  and constant  $\lambda$  such that

$$\frac{1}{2}\mathcal{L}_Vg = (r - \lambda)g \quad (2)$$

where  $\mathcal{L}_V$  is the Lie derivative operator along  $V$ . A Yamabe soliton is shrinking if  $\lambda < 0$ , steady if  $\lambda = 0$  and expanding if  $\lambda > 0$ , where  $\lambda$  is said to be soliton constant. Yamabe soliton has been studied by several authors in various contexts such as Blaga [4], Wang [5], Ghosh [6]. If  $\lambda$  is a smooth function then the so-called Yamabe solitons become almost Yamabe soliton.

According to Chen [3], Riemannian metric is called a  $k$ -almost Yamabe soliton if there exists a nonzero function  $k$ , a differentiable vector field  $V$ , and a differentiable function  $\lambda$  such that

$$\frac{k}{2}\mathcal{L}_Vg = (r - \lambda)g \quad (3)$$

When  $\lambda$  is constant, then the above equation becomes the k-Yamabe soliton.

From the above works, we motivats that to study  $k$ -Yamabe soliton in three-dimensional kenmotsu manifolds.

In the present paper, Kenmotsu 3-metric is considered as k-Yamabe soliton and some results are proved. The paper is organized as follows: in section 2, we remind some preliminary results of 3-dimensional Kenmotsu manifolds. In section 3, we proved that, if the flow vector field  $V$  is pointwise collinear with the Reeb vector field  $\xi$ , the vector field is constant multiple with  $\xi$ , the scalar curvature is constant and the soliton vector field is constant. Next, we prove that the manifold is the locally isometric to Hyperbolic space  $H2n+1(-1)$ . Finally, we prove that the manifold becomes Einstein manifold.

### 2. Preliminaries

A  $(2n+1)$  dimensional almost contact metric manifold  $M$  with an almost contact metric structure  $(\varphi, \xi, \eta, g)$  where  $\varphi$  is a  $(1, 1)$  tensor field,  $\xi$  is a vector field,  $\eta$  is a 1-form and  $g$  is the compatible Riemannian metric such that

$$\varphi^2X = -X + \eta(X)\xi, \eta(\xi) = 1, \varphi\xi = 0, \eta \circ \varphi = 0 \quad (4)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (5)$$

$$g(\varphi X, Y) = -g(X, \varphi Y) \quad (6)$$

$$g(X, \xi) = \eta(X) \quad (7)$$

for all vector fields  $X, Y$  on  $M$ . For details we refer to Blair [1]

An almost contact metric manifold is said to be a Kenmotsu manifold if

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$$(\nabla X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \quad (8)$$

for any vector fields X,Y on M. The following formulas are also satisfies for Kenmotsu manifold

$$\nabla X\xi = X - \eta(X)\xi \quad (9)$$

$$(\nabla X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \quad (10)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \quad (11)$$

From [11], we know that for a 3-dimensional Kenmotsu manifold

$$\begin{aligned} R(X, Y)Z &= \frac{r+4}{2}[g(Y, Z)X - g(X, Z)Y] - \\ &\frac{r+6}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \\ &\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \quad (12) \end{aligned}$$

and

$$S(X, Y) = \frac{1}{2}[(r + 2)g(X, Y) - (r + 6)\eta(X)\eta(Y)] \quad (13)$$

where, S is the ricci tensor, R is the curvature tensor and r is the scalar curvature of the manifold. Kenmotsu manifolds are studied by several authores such as Pitis [7], De, Yildiz and Yaliniz [8], Basu and Bhattacharyya [9], Kenmotsu [10].

### 3. Properties of 3-dimensional Kenmotsu manifold admitting k-Yamabe soliton:

**Theorem 1:** If a 3-dimensional Kenmotsu manifold M admits k-Yamabe soliton with the flow vector field V poitwise collinear with the Reeb vector field  $\xi$ , then V is a constant multiple of  $\xi$ , the scalar curvature is constant and V is killing.

**Proof.** Here we suppose that the vector field V is poitwise colliner with  $\xi$  ( $V=c \xi$ , where c is a smooth function on M). Then from (3)

$$k\xi c\xi g(X, Y) = 2(r - \lambda)g(X, Y)$$

which implies

$$k[g(\nabla Xc\xi, Y) + g(X, \nabla Y c\xi)] = 2(r - \lambda)g(X, Y) \quad (14)$$

Using(9) in (14), we get

$$k[cg(\nabla X\xi, Y) + (Xc)\eta(Y) + cg(X, \nabla Y \xi) + (Y c)\eta(X)] = 2(r - \lambda)g(X, Y) \quad (15)$$

Replacing Y by  $\xi$  in (15) gives,

$$k[(Xc) + (\xi c)\eta(X)] - 2(r - \lambda)\eta(X) = 0 \quad (16)$$

Again replacing X by  $\xi$  in (16) gives,

$$\xi c = \frac{r - \lambda}{k}$$

substituting the value of  $\xi c$  in (16), we get

$$dc = \frac{r - \lambda}{k} \eta \quad (17)$$

Applying d on (17) and using poincare lemma  $d^2 = 0$ , we get

$$\frac{r - \lambda}{k} d\eta = 0 \quad (18)$$

so, from (18) we say that

$$r = \lambda \quad (19)$$

using (19) in (17) gives

$$dc = 0$$

Then (3) yields

$$E_\xi g = 0 \quad (20)$$

i.e  $\xi$  is a killing vector field.

This complete the proof.

**Theorem 2:** If the metric of 3-dimensional Kenmotsu manifold M is a k-Yamabe soliton, then the manifold is locally isometric to the Hyperbolic space  $H^{(2n+1)}(-1)$

**Proof:** From (3), we have

$$\frac{k}{2}[g(\nabla_x V, Y) + g(X, \nabla_y V)] = (r - \lambda)g(X, Y) \quad (21)$$

for all X, Y and  $X= Y= \xi$  and V is orthogonal to  $\xi$  gives  $r= \lambda$

when  $X= \xi$  equation (9) gives

$$\nabla_\xi \xi = 0$$

From [5, Lemma 3.2], we know that  $r= \lambda= -6$ . Further it follows from (12) that

$$R(X, Y)Z = -g(Y, Z)X - g(X, Z)Y$$

for all the vector fields X, Y, Z. That means the constant sectional curvature is -1.

So, we can say that the manifold M 3 is locally isometric to the Hyperbolic space  $H^{2n+1}(-1)$

**Proposition-1:** 3- dimensional kenmotsu manifold admitting a k-Yamabe soliton is an Einstein manifold.

**Proof:** Ricci tensor of a 3-dimensional kenmotsu manifold is

$$\begin{aligned} S(X, Y) &= \frac{1}{2} [(r + 2)g(X, Y) \\ &- (r + 6)\eta(X)\eta(Y)] \end{aligned}$$

From theorem-1 we get,  $r = \lambda - 6$ .

If we use this value in the foregoing equation we get

$$S(X, Y) = -2g(X, Y)$$

Thus we conclude that k-Yamabe soliton  $(g, \xi, \lambda)$  on 3-dimensional Kenmotsu manifold is an Einstein manifold.

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