Research Article

Journal of Scientific Enquiry
Vol: 3 (1), Pages: 9-12

A note on fixed points of entire function of bicomplex variable

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Received: 05.04.2023; accepted: 26.05.2023; published online: 30.06.2023

Abstract

The prime concern of this paper is to derive the bi-complex analogue of the existence theorem of fixed points of order one of entire functions. We discuss some lemmas and prove the existence of an entire function $f(z)$ in $\mathbb{C}_2$ which has the given fixed points $a_1, a_2, \cdots$ with given multipliers $b_1, b_2, \cdots$ respectively, provided that the sequence $\{a_n\}$ has no finite limit point in $\mathbb{C}_2$. Finally, we give some examples having different kind of fixed points.

Keywords: Entire function, bi-complex variable, fixed points of order one, multiplier

AMS subject classification: $30D30, 30G35$

1. Introduction

The theory of bi-complex numbers is a matter of active research for quite a long time since the seminal work of Segre [5] in search of special algebra. The algebra of bi-complex numbers are widely used in the literature as it becomes a viable commutative alternative ([3], [4], [7], [8]) to the non-commutative skew field of quaternions introduced by Hamilton [3] (both are four-dimensional and generalization of complex numbers). We denote the complex plane by $\mathbb{C}_1$ throughout the paper. Consider two $\mathbb{C}_1$ functions u, v from $\mathbb{R}_2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ to $\mathbb{R}$. It is well known that if these two functions satisfy the so called Cauchy-Riemann system $\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}$, $\frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}$ then the function $f(x_1 + ix_2) = u(x_1, x_2) + iv(x_1, x_2)$ admits complex derivative. There are different ways to attempt to generalize this observation to the case of more pairs of real variables. To this purpose, we propose to complexify the Cauchy-Riemann system and to apply it two pairs of holomorphic functions u, v from $\mathbb{C}_2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}_1\}$ to $\mathbb{C}_1$ so that the pair (u, v) can be interpreted as a map of $\mathbb{C}_2$ to itself. It is then to think whether it makes any sense to consider the pairs (u, v) for which the following system is satisfied: $\frac{\partial u}{\partial z_1} = \frac{\partial v}{\partial z_2}$, $\frac{\partial u}{\partial z_2} = -\frac{\partial v}{\partial z_1}$.

Formally, we have replaced $\mathbb{R}$ by $\mathbb{C}_1$ and differentiability in the real sense by holomorphicity. As it turns out, it is possible to give a very interesting interpretation of this complexified Cauchy-Riemann system, if we endow the pair (z_1, z_2) with a special algebraic structure. Instead of considering (z_1, z_2) as a point in $\mathbb{C}_2$, we now consider in analogy with what we did in the case of $\mathbb{R}_2$, a new space where the elements are of the form $z = z_1 + jz_2$, where j is a new imaginary unit (i.e., $j^2 = -1$), which commuted with the original imaginary unit i. This creates a new algebra, the algebra of bi-complex numbers, defined by $\mathbb{C}_2 = \{z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}_1\}$ and as we will discuss in the next part of this paper, such an algebra enjoys most of the properties one would expect from a good generalization of the field of complex numbers. A bicomplex number is defined as $z = x_0 + ix_1 + jx_2 + ijx_3$, where x_0, x_1, x_2, x_3 are real numbers with $i^2 = j^2 = -1$ and $ij = ji$, $(ij)^2 = 1$, $\mathbb{C}_2$ becomes a real commutative algebra with identity $1 = 1 + i0 + j0 + ij0$, with standard binary composition.

So we can say that the bi-complex numbers are complex numbers with complex coefficients, which explains the name of bi-complex numbers.

Definition 1.1 *Given a bi-complex number $z = z_1 + jz_2$, its bi-complex conjugate is defined by $z' = z_1 - jz_2$.*

We immediately note that $zz' = z_1^2 + z_2^2 \in \mathbb{C}_1$. So in particular we can say that a bi-complex number $z = z_1 + jz_2$ is invertible if and only if $z_1^2 + z_2^2 \neq 0$ and the inverse of z is given by $z^{-1} = \frac{z'}{z_1^2 + z_2^2}$. If both z_1, z_2 are nonzero but the sum $z_1^2 + z_2^2$ is zero, then the corresponding bi-complex number $z = z_1 + jz_2$ is a zero divisor. In fact, all zero divisors in $\mathbb{C}_2$ are characterized by the equations $z_1^2 = -z_2^2$ i.e., $z_1 = \pm iz_2$. Thus, all zero divisors are of the form $z = \lambda(1 \pm ij)$, where $\lambda \in \mathbb{C} \setminus \{0\}$. The following definitions are immediate from the above discussion:

Definition 1.2 *There are two non trivial elements $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ in $\mathbb{C}_2$ with the properties $e_1^2 = e_1, e_2^2 = e_2, e_1 e_2 = e_2 e_1 = 0$ and $e_1 + e_2 = 1$ which are called the idempotent elements in $\mathbb{C}_2$.*

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Definition 1.3 *Any bi-complex number $z = z_1 + jz_2$ can be written as $z = \alpha e_2 + \beta e_2$ where $\alpha = z_1 - iz_2$ and $\beta = z_1 + iz_2$ are uniquely defined complex numbers. This is known as the idempotent representation of z.*

These results a pair of mutually complementary projections: $P_1: (z_1 + jz_2) \in \mathbb{C}_2 \mapsto (z_1 - jz_1) \in \mathbb{C}_1$ and $P_2 : (z_1 + jz_2) \in \mathbb{C}_2 \mapsto (z_1 + jz_1) \in \mathbb{C}_1$.

Definition 1.4 *An element $z = z_1 + jz_2$ is singular if and only if $|z_1^2 + z_2^2| = 0$. The set of singular elements is denoted as O_2 and is defined by $O_2 = \{z \in \mathbb{C}_2 : z$ is the collection of all complex multiple of e_1 and $e_2\}$.*

Definition 1.5 *The norm $\|.\|: \mathbb{C}_2 \mapsto \mathbb{R}^+ \cup \{0\}$ of a bi-complex number is defined as $\| z \| = \sqrt{\{|z_1^2| + |z_2^2|\}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.*

1.1 Bi-complex function

We start with a bi-complex valued function $f : \Omega \subset \mathbb{C}_2 \to \mathbb{C}_2$. The derivative of f at a point $z_0 \in \Omega$ is defined by $f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ provided the limit exists and the domain is so chosen that $h = h_0 + ih_1 + jh_2 + ijh_3$ is invertible. It is easy to prove that h is not invertible only for $h_0 = -h_3, h_1 = h_2$ or $h_0 = h_3, h_1 = -h_2$. If the bi-complex derivative of f exists at each point of its domain Ω then in similar to complex function, f will be a bi-complex holomorphic function in Ω. Indeed, if f can be expressed as $f(z) = g_1(z_1, z_2) + jg_2(z_1, z_2)$, where $z = z_1 + jz_2 \in \Omega$, then f will be holomorphic if and only if g_1, g_2 are both complex holomorphic in z_1, z_2 and $\frac{\partial g_1}{\partial z_1} = \frac{\partial g_2}{\partial z_2}, \frac{\partial g_1}{\partial z_2} = -\frac{\partial g_2}{\partial z_1}$. Moreover, $f'(z) = \frac{\partial g_1}{\partial z_2} + j\frac{\partial g2}{\partial z1}$ and it is invertible only when $\det\begin{pmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} \end{pmatrix} \neq 0$. Let $f: X \to \mathbb{C}_2$ where $X \subseteq \mathbb{C}_2$ be a bi-complex valued holomorphic function. Now the fixed points of $f(z)$ are the points z' for which $f(z') = z'$. Now, since $f(z)$ is bi-complex valued, so $f(z) = f_1(\alpha)e_1 + f_2(\beta)e_2$, where $\alpha = z_1 - iz_2, \beta = z_1 + iz_2$ and $z = \alpha e_1 + \beta e_2 = z_1 + jz_2$. Now, $f(z') = z'$ implies that $f_1(\alpha')e_1 + f_2(\beta')e_2 = \alpha' e_1 + \beta' e_2$. This gives us that, $f_1(\alpha') = \alpha'$ and $f_2(\beta') = \beta'$. So, the fixed points of $f(z)$ implies the fixed points of f_1 and f_2 at some points, where f_1 and f_2 are complex valued functions. The existence and distribution of the fixed points of entire functions $f(z)$ of the bi-complex variable z are important in the study of iteration of these functions.

Definition 1.6 *The number $f'(z') \equiv \frac{d}{dz} f(z)$ at $z = z'$ is called the multiplier of z'.*

For integral $n > 0$, the iterates of $f(z)$ are defined by, $z_1 = f(z), z_n = f(z_{n-1}) = f_n(z) = f_{n-1}(f(z))$. Thus, $z_1 = f(z) = (f_1 e_1 + f_2 e_2)(z) = (f_1 e_1 + f_2 e_2)(\alpha e_1 + \beta e_2) = f_1(\alpha)e_1 + f_2(\beta)e_2$ and $z_2 = f(z_1) = f(f(z)) = f_2(z) = (f_1 e_1 + f_2 e_2)((f_1 e_1 + 2e_2)(\alpha e_1 + \beta e_2)) = (f_1^2 e_1 + f_2^2 e_2)(\alpha e_1 + \beta e_2) = f_1^2(\alpha)e_1 + f_2^2(\beta)e_2$. Similarly, $z_n = f(z_{n-1}) = f_1^n(\alpha)e_1 + f_2^n(\beta)e_2$. Thus iterates of positive integral order of entire functions satisfying the functional equations:

$$f_m(f_n(z)) = f_n(f_m(z)) = f_{m+n}(z). \qquad (1)$$

Definition 1.7 *For integral $p > 0$, the fixed points of order p of $f(z)$ are those points z' for which $f^p(z') = z'$ where, $f^k(z') \neq z'$ for $k < p$.*

Thus fixed points of order one of $f(z)$ are those points z' for which $f_1(z') \equiv f(z') = z'$, which is already defined. Next we classify fixed points of order p of $f(z)$ in the following manner and the following definition is immediate:

Definition 1.8 *The fixed points of order $p(p > 0$ being an integer) of $f(z)$ are classified as attractive, indifferent or repulsive according as their modulii of multipliers $f_p'(z')$ are $< 1, = 1$ or > 1.*

In this paper our main target is to establish the existence theorem of fixed points of order one of entire function under the flavour of bi-complex analysis. We do not explain the standard theories, definitions and notations of bi-complex analysis as those are available in [2], [3], [4], [5], [6].

2. Some lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 *If $\{z_n\}$ is an arbitrary sequence of complex numbers different from zero and whose sole limit point is ∞ and if m is a non negative integer, then there exist an entire function $G(z)$ having roots at the points $z1, z2, \cdots$ (and these points only) and a root of multiplicity m at the point zero. Further, $G(z)$ can be defined by the absolutely uniformly convergent product*

$$G(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{Q_\nu\left(\frac{z}{z_n}\right)},$$

where $e^{g(z)}$ is an arbitrary entire function and $Q_\nu(z)$ is a polynomial such that

$$Q_\nu(z) = z + \frac{z^2}{2} + \cdots + \frac{z^\nu}{\nu}.$$

The non negative integer ν has the property that the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\nu+1}}$ converges uniformly in the whole complex plane.

Remark 2.1 *Lemma 2.1 is known as Weierstrass's factorization theorem of entire functions in the theory of complex analysis.*

The next lemma is the bi-complex analogue of Lemma 2.1.

Lemma 2.2 *[1] Let $z_1, z_2, z_3, \ldots$ be any sequence of bi-complex numbers and infinity be the only limit point of it. Then it is possible to construct an entire function of bi-complex variable which vanishes at each of these points z_n.*

3. Main results

In this section, we present the main result of the paper. The following theorem ensures the existence of fixed points of order one of a bi-complex valued function.

Theorem 3.1 *There exists an entire function $f(z)$ in $\mathbb{C}_2$ which has the given fixed points $a_1, a_2, \ldots$ with given multipliers $b_1, b_2, \ldots$ respectively, provided that the sequence $\{a_n\}$ has no finite limit point in $\mathbb{C}_2$.*

Proof. Since $f(z)$ is a bi-complex valued function, therefore $f(z)$ can be expressed as $f(z) = f_1(\alpha)e_1 + f_2(\beta)e_2$ where $z = \alpha e_1 + \beta e_2$. Also, a_i and b_i for $i = 1, 2, 3, \ldots$ are all bi-complex numbers, therefore $a_i = a_i'^{e_1} + a_i'' e_2$ and $b_i = b_i' e_1 + b_i'' e_2$ for $i = 1, 2, \ldots$. First let $h(z)$ be an entire function whose zeros are precisely the points an and such that a_n is a simple zero if $b_n \neq 1$ and a double zero if $b_n = 1$ so that, $h'(a_n) \neq 0$ if $b_n \neq 1$ while $h'(a_n) = 0$ if $b_n = 1$. Now, $h(a_n) = 0$ implies that $h(a_n' e_1 + a_n'' e_2) = 0$. That is $h_1(a_n')e_1 + h_2(a_n'')e_2 = 0 = 0e_1 + 0e_2$. Again this implies $h_1(a_n') = 0$ and $h_2(a_n'') = 0$. So what we need to prove is $f'(a_n) = b_n$ if $b_n \neq 1$. That is $f_1'(a_n')e_1 + f_2'(a_n'')e_2 = b_n' e_1 + b_n'' e_2$ if $b_n' e_1 + b_n'' e_2 \neq 1 = e_1 + e_2$. This implies $f_1'(a_n') = b_n', f_2'(a_n'') b_n''$ if $b_n' \neq 1, b_n'' \neq 1$. Also we need to prove $f'(a_n) = 1$ if $b_n = 1$. That is $f_1'(a_n')e_1 + f_2'(a_n'')e_2 = e_1 + e_2$ if $b_n' e_1 + b_n'' e_2 = 1 = e_1 + e_2$. This implies that $f_1'(a_n') = 1, f_2'(a_n'') = 1$ if $b_n' = 1, b_n'' = 1$. As the existence of fixed points of order one of f implies the same for f_1 and f_2 so $h(z) = h_1(\alpha)e_1 + h_2(\beta)e_2$ may be constructed by Weierstrass's factorization theorem for entire functions. Let $k(z) = k_1(\alpha)e_1 + k_2(\beta)e_2$ be an bi-complex valued entire function such that

$$\begin{aligned} k_1(a_n') &= \log\left[\frac{b_n'-1}{h_1'(a_n')}\right] \text{ if } b_n' \neq 1 \\ &= 1 \text{ if } b_n' = 1 \end{aligned}$$

and

$$\begin{aligned} k_2(a_n'') &= \log\left[\frac{b_n''-1}{h_1'(a_n'')}\right] \text{ if } b_n'' \neq 1 \\ &= 1 \text{ if } b_n' = 1 \end{aligned}$$

where any determination of the logarithm may be taken. Now let us put $f_1(\alpha) = \alpha + h_1(\alpha)\, exp\{k_1(\alpha)\}$ and $f_2(\beta) = \beta + h_2(\beta)\, exp\{k_2(\beta)\}$.

Clearly, the fixed points of $f_1(\alpha)$ and $f_2(\beta)$ are the zeros of $h_1(\alpha)$ and $h_2(\beta)$ respectively. Let us now examine the multipliers of a_n' and a_n''. We have $f_1'(\alpha) = 1 + h_1'(\alpha)\, exp\{k_1(\alpha)\} + h_1(\alpha) exp\{k_1(\alpha)\}\, k_1'(\alpha)$. This gives that $f_1'(a_n') = 1 + h_1'(a_n')\, exp\{k_1(a_n')\} + h_1(a_n')\, exp\{k_1(a_n')\} k_1'(a_n')$. Similarly $f_1'(a_n'') = 1 + h_1'(a_n'') exp\{k_1(a_n'')\} + h_1(a_n'')\, exp\{k_1(a_n'')\} k_1'(a_n'')$. So, if $b_n \neq 1$, i.e., $b_n' \neq 1$ and $b_n'' \neq 1$, we have,

$$\begin{aligned} f_1'(a_n') &= 1 + h_1'(a_n')\left[\frac{b_n'-1}{h_1'(a_n')}\right] + 0\left[\frac{b_n'-1}{h_1'(a_n')}\right] k_1'(a_n') \\ &= b_n', \text{ since, } h_1'(a_n') \neq 0. \end{aligned}$$

Like wise $f_2'(a_n'') = b_n''$. Again if $b_n = 1$, we have, $f_1'(a_n') = 1 + h_1'(a_n')0 + 0\, exp\{k_1(a_n')\} = 1$, since $h_1'(a_n') = 0$. In a like manner $f_2'(a_n'') = 1$. Therefore, the multiplier of a_n is

$$\begin{aligned} f'(a_n) &= (f_1' e_1 + f_2' e_2)(a_n) \\ &= f_1'(a_n')e_1 + f_2'(a_n'')e_2 \\ &= b_n' e_1 + b_n'' e_2, \text{ if } b_n', b_n'' \neq 1 \\ &= b_n \end{aligned}$$

and

$$\begin{aligned} f'(a_n) &= (f_1' e_1 + f_2' e_2)(a_n) \\ &= f_1'(a_n')e_1 + f_2'(a_n'')e_2 \\ &= e_1 + e_2, \text{ if } b_n', b_n'' = 1 \\ &= 1. \end{aligned}$$

Hence,

$$\begin{aligned} f'(a_n) &= b_n, \text{ if } b_n \neq 1 \\ &= 1, \text{ if } b_n = 1. \end{aligned}$$

This proves the theorem. ∎

Remark 3.1 *The converse of Theorem 3.1 is not true, which is evident from the following example.* Let $f(z) = e^z + z$ then $e^z + z = z$ implies $e^z = 0$, which is impossible. Hence, $e^z + z$ has no fixed point.

Remark 3.2 *Consider the function*

$$f(z) = z - [g(z)]^2,$$

where $g(z)$ is an entire function having infinitely many zeros. It can be shown that $f(z)$ has no attractive fixed point, an infinite number of indifferent fixed points at the zeros of $g(z)$ and no repulsive fixed point.

Remark 3.3 *If we take the function*

$$f(z) = z + ze^z,$$

we see that, $f(z)$ has one repulsive fixed point at $z = 0$ with multiplier 2 and no attractive or indifferent fixed point.

Future scope: In the line of Theorem 3.1, one may think of the derivation of existence of fixed points of order p ($p > 0$, being an integer) of a bicomplex valued entire function which may be considered as an open problem in this area.

Conflict of Interest: The authors do not have any conflicts of interest.

Acknowledgement: The authors express their heartfelt thanks to the reviewer(s) for careful reading of our paper and helpful comments and suggestions for the betterment of the paper. The third author sincerely acknowledges the financial support of the Post Seed Grant received from Sidho-Kanho-Birsha University, Purulia-723104, India [Ref No.: $R/847/SKBU/2022$; Date: 20.07.2022] and the fourth author sincerely acknowledges the financial support rendered by DST-FIST running at the Department of Mathematics, University of Kalyani, Kalyani-741235, India.

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