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A Note on (p,q)-φ order and (p,q)-φ type of a Meromorphic function

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Abstract: In the paper, firstly we present several new results on the equality of (p,q)-φ order (lower order) of a meromorphic function and those of its derivative. Next, we study the relationship between (p,q)-φ order (lower order), (p,q)-φ type of a meromorphic function and those of a differential monomial generated by the function, where p, q are positive integers satisfying $p \geq q \geq 1$ and φ (r): $[0, \infty) \rightarrow (0,\infty)$ is a non-decreasing unbounded function of r.
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1 Introduction, Definitions and Notations

The study of relationship among the various properties of entire and meromorphic function and their derivatives is one of the important characteristics of the Value distribution theory. Since the beginning, the order, type of entire and meromorphic function and those of their derivatives are always a topic of greater interest among researchers.

In 1949, Valiron [13] presented the well-known result that shows the equality of order of an entire function and that of its derivative, whereas Tsuji [12], Whittaker [14] and many more proved the same for a meromorphic function.

In the year 1988 and 1990, Lahiri [7,8] established various results on the equality of generalised order (lower order) of a meromorphic function and that of its derivative.

Here, the standard notations and definitions of Value distribution theory due to R. Nevanlinna such as Characteristic function $T(r,f)$, Enumerative function $N(r,f)$, Proximity function $m(r,f)$, order ρ_f, type σ_f etc. are not described, because one may easily read from [5, 13, 15] and many more. We use the following notation often in the sequel:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$$

for k=1,2,3,... and $\log^{[0]} x = x$.

In the year 1976, Juneja et al. [6] have introduced the definition of (p,q)-order and (p,q)-lower order for an entire function. After modifying the original definition in 2012, Li and Cao [10] defined (p,q)-order and (p,q)-lower order for a meromorphic function.

Later in the year 2014, Shen et al. [11] initiated the concept of (p,q)-φ order and (p,q)-φ lower order for a meromorphic function, where p,q are positive integers satisfying $p \geq q \geq 1$ and $\varphi(r) : [0, \infty \rightarrow (0, \infty)$ is a non-decreasing unbounded function of r.

All the earlier definitions framed by the various researchers time to time, such as

classical order, generalised k-order (or iterated k-order), standard (p,q)-order etc. had certain limitations since they are unable to cover every possible growth rate of an entire or a meromorphic function. The introduction of the more general (p,q)-φ order addresses this weakness, providing a more versatile growth indicator that covers a wide range of growth rates.
Here, in the below, we state the definitions introduced by Shen et al. [11].

Definition 1.1 [11] The (p,q)-φ order and (p,q)-φ lower order of a meromorphic function f is defined respectively as follows:

$$\rho_{p,q}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f)}{\log^{[q]}\varphi(r)}$$

$$\lambda_{p,q}(f,\varphi)=\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f)}{\log^{[q]}\varphi(r)}$$

where p, q are positive integers satisfying $p\geq q\geq 1$.
Further Shen et al. [11] imposed another two conditions on $\varphi(r)$ given as below

$$(i)\quad \lim_{r\to\infty}\frac{\log^{[p+1]}r}{\log^{[q]}\varphi(r)}=0 \quad and$$

$$(ii)\quad \lim_{r\to\infty}\frac{\log^{[q]}\varphi(\alpha r)}{\log^{[q]}\varphi(r)}=1\ for\ some\ \alpha>1$$

and proposed the following definition:
Definition 1.2 [11] The (p,q)-φ order and (p,q)-φ lower order of an entire function $f$is defined respectively as follows:

$$\rho_{p,q}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f)}{\log^{[q]}\varphi(r)}=\overline{\lim_{r\to\infty}}\frac{\log^{[p+1]}M(r,f)}{\log^{[q]}\varphi(r)}$$

$$\lambda_{p,q}(f,\varphi)=\underline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f)}{\log^{[q]}\varphi(r)}=\underline{\lim_{r\to\infty}}\frac{\log^{[p+1]}M(r,f)}{\log^{[q]}\varphi(r)}$$

where p, q are positive integers satisfying $p\geq q\geq 1$.

Remark 1.1 If we take p=q=1 and $\varphi(r)=r$, then $\rho_{(1,1)}(f,\varphi)$ and $\lambda_{(1,1)}(f,\varphi)$ coincide with the definition of classical order ρ_f and lower order λ_f respectively.

In line with the above definitions, one may also define the (p,q)-φ type for entire and meromorphic function respectively as follows:
Definition 1.3 The (p,q)-φ type for a meromorphic function f is defined as follows:

$$\sigma_{(p,q)}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}T(r,f)}{\left[\log^{[q-1]}\varphi(r)\right]\rho_{(p,q)}(f,\varphi)},$$

$0<\rho_{(p,q)}(f,\varphi)<\infty$.
If f is an entire function, then

$$\sigma_{(p,q)}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}M(r,f)}{\left[\log^{[q-1]}\varphi(r)\right]\rho_{(p,q)}(f,\varphi)},$$

$0<\rho_{(p,q)}(f,\varphi)<\infty$
where p, q are positive integers satisfying $p\geq q\geq 1$.
Remark 1.2. If we take p=q=1 and $\varphi(r)=r$, then $\sigma_{(1,1)}(f,\varphi)$ coincides with the definition of classical type σ_f.
Here, we prove some results related to the equality of (p,q)-φ order (lower order) of a meromorphic function and those of its derivative, where p, q are positive integers satisfying $p\geq q\geq 1$ and $\varphi(r):[0,\infty)\to(0,\infty)$ is a non-decreasing unbounded function of r.
Next, we consider a transcendental meromorphic function f defined in the open complex plane $\mathbb{C}$. We write the differential monomial generated by f as follows:

$$P\left[f\right]=Af^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$$

where $n_0,n_1,...,n_k$ be the non-negative integers such that $\sum_{i=0}^{k}n_i\geq 1$ and $T(r,A)=o\left\{T(r,f)\right\}$ as $r\to\infty$ possibly outside a set E of finite linear measure.

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The numbers $\gamma_P = \sum_{i=0}^{k} n_i$ and $\Gamma_P = \sum_{i=0}^{k}(i+1)n_i$ are called the degree and weight of $P[f]$ respectively [9].

Definition 1.5 [16]. For $a \in \mathbb{C} \cup \{\infty\}$, we denote the number of simple zeros of $f - a$ in $|z| \leq r$ by $n(r, a; f \;|= 1)$. Here $N(r, a; f \;|= 1)$ is defined in terms of $n(r, a; f \;|= 1)$ in the usual way. We write the deficiency of a corresponding to the simple a–points of f, i.e., simple zeros of $f - a$, as

$$\delta_1(a; f) = 1 - \overline{\lim_{r\to\infty}} \frac{N(r, a; f \;|= 1)}{T(r, f)}.$$

Further, Yang [15] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) \leq 4$.

In the year 2008, Datta and Kar [2] studied the growth properties of a differential monomial generated by a meromorphic function and proved some results showing the relationship between the order (lower order), type of a meromorphic function and those of a differential monomial generated by the function.
Here we present newly developed results based on the relationship between (p, q)–φ order (lower order), (p, q)–φ type of a meromorphic function and those of a differential monomial generated by the function, where p, q are positive integers satisfying $p \geq q \geq 1$ and $\varphi(r)\colon [0, \infty) \to (0, \infty)$ is a non-decreasing unbounded function of r.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.
Lemma 2.1 [8]. Let f be a transcendental meromorphic function, then for all large values of r,

$$T(r, f') \leq 2T(2r, f) + o\{T(2r, f)\}.$$

Lemma 2.2 [15,1]. Let f be a meromorphic function, then for all large values of r,

$$T(r, f) < c\{T(2r, f') + \log r\},$$

where c is a constant depending only on $f(0)$.
Lemma 2.3 [3]. Let f be a transcendental meromorphic function of finite order, having finite number of zeros, satisfying the conditions $f(0) \neq 0, \infty$ and $f'(0) \neq 0$, then

(i) $T(r, f') \leq \{2 + o(1)\}T(r, f)$ as $r \to \infty$, *and*
(ii) $\{1 + o(1)\}T(r, f) \leq T(r, f')$ as $r \to \infty$.

Lemma 2.4 [9]. Let f be a transcendental meromorphic function of finite order or of non-zero lower order, and satisfies the condition $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$, then

$$\lim_{r\to\infty} \frac{T(r, P[f])}{T(r, f)} = \Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f),$$

where

$$\Theta(\infty; f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, f)}{T(r, f)}.$$

3 Main Results

In this section we present the main results of the paper as below.

Theorem 3.1. Let f be a meromorphic function, then the (p, q)–ϕ order (lower order) of the function and those of its derivative f' are same, i.e.,

$$\rho_{(p,q)}(f, \varphi) = \rho_{(p,q)}(f', \varphi) \quad \text{and}$$

$$\lambda_{(p,q)}(f, \varphi) = \lambda_{(p,q)}(f', \varphi),$$

where p, q are positive integers satisfying $p \geq q \geq 1$ and $\varphi(r)\colon [0, \infty) \to (0, \infty)$ is a non-decreasing unbounded function of r.

Proof. By Lemma 2.1, for all large values of r, we have

$$T(r, f') \leq 2T(2r, f) + o\{T(2r, f)\},$$

from which we obtain

$$\log^{[p]} T(r, f') \leq \log^{[p]} T(2r, f) + O(1).$$

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Since $\varphi(r)$ satisfies the condition

$$\lim_{r\to\infty}\frac{\log^{[q]}\varphi(\alpha r)}{\log^{[q]}\varphi(r)}=1 \quad \text{for some } \alpha>1,$$

we have from the above

$$\begin{aligned}\rho_{(p,q)}(f',\varphi)&=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f')}{\log^{[q]}\varphi(r)}\\ &\le \overline{\lim_{r\to\infty}}\left\{\frac{\log^{[p]}T(2r,f)}{\log^{[q]}\varphi(2r)}\cdot\frac{\log^{[q]}\varphi(2r)}{\log^{[q]}\varphi(r)}\right\}\\ &=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(2r,f)}{\log^{[q]}\varphi(2r)}\cdot\lim_{r\to\infty}\frac{\log^{[q]}\varphi(2r)}{\log^{[q]}\varphi(r)}\\ &=\rho_{(p,q)}(f,\varphi). \qquad ...(3.1)\end{aligned}$$

Again, by Lemma 2.2, for all large values of r, we have

$$T(r,f)<c\{T(2r,f')+\log r\},$$

where c is a constant depending only on $f(0)$. Also f being transcendental, we have

$$\log r=o\{T(r,f)\}.$$

Now from the above we can easily get

$$\log^{[p]}T(r,f)+O(1)\le\log^{[p]}T(2r,f'),$$

from which we obtain

$$\begin{aligned}\rho_{(p,q)}(f,\varphi)&=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f)}{\log^{[q]}\phi(r)}\\ &\le \overline{\lim_{r\to\infty}}\left\{\frac{\log^{[p]}T(2r,f')}{\log^{[q]}\phi(2r)}\cdot\frac{\log^{[q]}\phi(2r)}{\log^{[q]}\phi(r)}\right\}\\ &=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(2r,f')}{\log^{[q]}\phi(2r)}\cdot\lim_{r\to\infty}\frac{\log^{[q]}\phi(2r)}{\log^{[q]}\phi(r)}\\ &=\rho_{(p,q)}(f',\phi). \qquad ...(3.2)\end{aligned}$$

Combining (3.1) and (3.2), we have

$$\rho_{(p,q)}(f,\varphi)=\rho_{(p,q)}(f',\varphi).$$

Proceeding similarly as above, one can easily prove the following:

$$\lambda_{(p,q)}(f,\varphi)=\lambda_{(p,q)}(f',\varphi).$$

This proves the theorem.

Remark 3.1. Theorem 3.1 is analogous to Theorem 1 [8] or somewhat may be treated as an improvement in the sense of further generalisation since two parameters p,q $(p\ge q\ge 1)$ are being considered in the present Theorem 3.1, whereas in Theorem 1 [8], Lahiri considered only one parameter k (≥ 2) while proving the equality of generalised order (lower order) of a meromorphic function and that of its derivative. Moreover, the presence of the function $\varphi(r)$ gives the flexibility of choosing specific functions for certain special cases.

We give the following example which validates Theorem 3.1.

Example 3.1. Let us consider the function $f(z)=\exp(z^2)$.
For $p=q=1$, it can be easily checked that

$$\rho_{(p,q)}(f,\varphi)=\rho_{(p,q)}(f',\varphi)=2 \quad\text{and}$$
$$\lambda_{(p,q)}(f,\varphi)=\lambda_{(p,q)}(f',\varphi)=2,$$

where $\varphi(r)=r\exp(1/r)$.

Theorem 3.2. Let f be a transcendental meromorphic function of finite order, having finite number of zeros, satisfies the conditions $f(0)\ne 0,\infty$; $f'(0)\ne 0$, then the (p,q)–φ order (lower order) of the function and those of its derivative f' are same, i.e.,

$$\rho_{(p,q)}(f,\varphi)=\rho_{(p,q)}(f',\phi) \quad\text{and}$$
$$\lambda_{(p,q)}(f,\varphi)=\lambda_{(p,q)}(f',\varphi),$$

where p,q are positive integers satisfying $p\ge q\ge 1$ and $\phi(r)\colon[0,\infty)\to(0,\infty)$ is a non-decreasing unbounded function of r.

Proof. By the result (i) of Lemma 2.3, we have

$$T(r,f')\le\{2+o(1)\}T(r,f)\quad\text{as } r\to\infty,$$

from which we get the following

$$\log^{[p]}T(r,f')\le\log^{[p]}T(r,f)+O(1).$$

$$\begin{aligned}\therefore\ \rho_{(p,q)}(f',\varphi)&=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f')}{\log^{[q]}\varphi(r)}\\ &\le\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T(r,f)}{\log^{[q]}\varphi(r)}\\ &=\rho_{(p,q)}(f,\varphi) \quad ...(3.3)\end{aligned}$$

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Also, by the result (ii) of Lemma 2.3, we have

$$\{1+o(1)\}T(r,f) \leq T(r,f') \quad \text{as } r \to \infty,$$

from which we obtain

$$\log^{[p]} T(r,f) + O(1) \leq \log^{[p]} T(r,f').$$

Therefore

$$\begin{aligned}\rho_{(p,q)}(f,\varphi) &= \varlimsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} \varphi(r)} \\ &\leq \varlimsup_{r\to\infty} \frac{\log^{[p]} T(r,f')}{\log^{[q]} \varphi(r)} \\ &= \rho_{(p,q)}(f',\varphi). \quad ...(3.4)\end{aligned}$$

Combining (3.3) and (3.4), we have

$$\rho_{(p,q)}(f,\phi) = \rho_{(p,q)}(f',\varphi).$$

Similarly proceeding as before, one can easily prove the following

$$\lambda_{(p,q)}(f,\varphi) = \lambda_{(p,q)}(f',\varphi).$$

Hence the theorem is proved.

Remark 3.2. Theorem 3.2 is analogous to Theorem 2 [7] or somewhat may be treated as an improvement in the sense of further generalisation since two parameters p, q $(p \geq q \geq 1)$ are being considered in the present Theorem 3.2, whereas in Theorem 2 [7], Lahiri considered only one parameter k (≥ 2) while proving the equality of iterated order of a meromorphic function and that of its derivative. Moreover, the presence of the function $\phi(r)$ gives the flexibility of choosing specific functions for certain special cases.

Theorem 3.3. Let f be a transcendental meromorphic function of finite order or of non-zero lower order, and satisfies the condition $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, then the (p,q)–ϕ order (lower order) of the function and those of the differential monomial $P[f]$ generated by the function are same. i.e.,

$$\rho_{(p,q)}(f,\varphi) = \rho_{(p,q)}(P[f],\varphi) \quad \text{and}$$
$$\lambda_{(p,q)}(f,\varphi) = \lambda_{(p,q)}(P[f],\varphi),$$

where p, q are positive integers satisfying $p \geq q \geq 1$ and $\varphi(r)\colon [0,\infty) \to (0,\infty)$ is a non-decreasing unbounded function of r.

Proof. By Lemma 2.4, we have

$$\lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)} = \Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty;f),$$

where

$$\Theta(\infty;f) = 1 - \varlimsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)}.$$

From the above, we can easily prove the following result

$$\lim_{r\to\infty} \frac{\log^{[p]} T(r,P[f])}{\log^{[p]} T(r,f)} = 1.$$

Now, using the above result we have

$$\begin{aligned}\rho_{(p,q)}(P[f],\varphi) &= \varlimsup_{r\to\infty} \frac{\log^{[p]} T(r,P[f])}{\log^{[q]} \varphi(r)} \\ &= \varlimsup_{r\to\infty} \left\{ \frac{\log^{[p]} T(r,f)}{\log^{[q]} \varphi(r)} \cdot \frac{\log^{[p]} T(r,P[f])}{\log^{[p]} T(r,f)} \right\} \\ &= \varlimsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} \varphi(r)} \cdot \lim_{r\to\infty} \frac{\log^{[p]} T(r,P[f])}{\log^{[p]} T(r,f)} \\ &= \rho_{(p,q)}(f,\varphi).\end{aligned}$$

Proceeding in a similar way, one can easily prove the following

$$\lambda_{(p,q)}(f,\varphi) = \lambda_{(p,q)}(P[f],\varphi).$$

Hence the theorem is proved.

Theorem 3.4. Let f be a transcendental meromorphic function of finite order or of non-zero lower order, and satisfies the condition $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$, then for $0 < \rho_{(p,q)}(f,\varphi) < \infty$, the following holds:

$$\sigma_{(p,q)}(P[f],\varphi) = \sigma_{(p,q)}(f,\varphi),$$

where p, q are positive integers satisfying $p \geq q \geq 1$ $(p \neq 1)$ and $\varphi(r)\colon [0,\infty) \to (0,\infty)$ is a non-decreasing unbounded function of r.

Proof. From Lemma 2.4, we get

$$\lim_{r\to\infty} \frac{T(r,P[f])}{T(r,f)} = \Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty;f),$$

where

$$\Theta(\infty;f) = 1 - \varlimsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)}.$$

The following result can be easily proved from the above:

$$\lim_{r\to\infty} \frac{\log^{[p]} T(r,P[f])}{\log^{[p]} T(r,f)} = 1.$$

Now, using the above result we have

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$$\begin{aligned}
&\sigma_{(p,q)}(P[f],\varphi)\\
&=\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}T(r,P[f])}{(\log^{[q-1]}\varphi(r))^{\rho_{(p,q)}(P[f],\varphi)}}\\
&=\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}T(r,P[f])}{(\log^{[q-1]}\varphi(r))^{\rho_{(p,q)}(f,\varphi)}}\\
&=\overline{\lim_{r\to\infty}}\left\{\frac{\log^{[p-1]}T(r,f)}{(\log^{[q-1]}\varphi(r))^{\rho_{(p,q)}(f,\varphi)}}\cdot\frac{\log^{[p-1]}T(r,P[f])}{\log^{[p-1]}T(r,f)}\right\}\\
&=\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}T(r,f)}{(\log^{[q-1]}\varphi(r))^{\rho_{(p,q)}(f,\varphi)}}\cdot\lim_{r\to\infty}\frac{\log^{[p-1]}T(r,P[f])}{\log^{[p-1]}T(r,f)}\\
&=\sigma_{(p,q)}(f,\varphi).
\end{aligned}$$

This proves the theorem.

Corollary 3.1. If we consider $p=q=1$, then under the similar conditions as stated in Theorem 3.4, the relationship between the (p,q)–ϕ type of f and that of $P[f]$ takes the form as given below:

$$\begin{aligned}
&\sigma_{(1,1)}(P[f],\phi)\\
&=\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\}\,\sigma_{(1,1)}(f,\varphi),
\end{aligned}$$

where

$$\Theta(\infty;f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}.$$

Proof. By Lemma 2.4, we have

$$\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}=\Gamma_P-(\gamma_P-\gamma_P)\Theta(\infty;f)$$

Now, from the definition, we obtain

$$\begin{aligned}
&\sigma_{(1,1)}(P[f],\varphi)=\overline{\lim_{r\to\infty}}\frac{T(r,P[f])}{[\varphi(r)]^{\rho_{(1,1)}(P[f],\varphi)}}\\
&=\overline{\lim_{r\to\infty}}\frac{T(r,P[f])}{[\varphi(r)]^{\rho_{(1,1)}(f,\varphi)}}\\
&=\overline{\lim_{r\to\infty}}\left\{\frac{T(r,f)}{[\varphi(r)]^{\rho_{(1,1)}(f,\varphi)}}\cdot\frac{T(r,P[f])}{T(r,f)}\right\}\\
&=\overline{\lim_{r\to\infty}}\frac{T(r,f)}{[\varphi(r)]^{\rho_{(1,1)}(f,\varphi)}}\cdot\lim_{r\to\infty}\frac{T(r,P[f])}{T(r,f)}\\
&=\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\}\,\sigma_{(1,1)}(f,\varphi).
\end{aligned}$$

Hence the result is proved.

Remark 3.3 Theorem 3.3 and Theorem 3.4 are the generalisation of the Lemma 4 [2].

4 Conclusion

Throughout the paper, during the study of comparing the (p,q)-φ order (lower order), (p,q)-φ type between a meromorphic function and its derivative or a differential monomial generated by the function, it can be noticed easily that $\varphi(r)$ must be non-decreasing and unbounded, moreover, it has to satisfy certain limiting conditions too. Apart from that p and q must be two positive integers satisfying the condition $p\geq q\geq 1$. All the results developed so far in the paper are based on the above restrictions. So, one may try for further improvement, by relaxing any of these conditions from either of the part of p, q or $\varphi(r)$. Further, in the course of journey of comparing growth properties, we have considered derivative of the function as well as differential monomial, but one step ahead, any researcher may go for comparison with differential polynomial and the outcome may be surprising. This leads to the direction of the future study.

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Conflicts of Interest

The author declares no conflicts of interest regarding this manuscript.

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