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EXTENSIONS, NUMBERS AND FREGE'S VISION OF LOGIC AS A UNIVERSAL LANGUAGE

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Gottlob Frege was a German mathematician, logician, and philosopher who is widely regarded as one of the founders of modern logic and analytic philosophy. Frege's major contributions were in the areas of logic and the philosophy of language. He sought to provide a formal foundation for mathematics by developing a logical system that could capture the basic principles and concepts of arithmetic. His groundbreaking work in this area is presented in his book *'The Foundations of Arithmetic'*. Paragraph 64 from *Foundations of Arithmetic* contains one of Frege's most famous assertions. It is the assertion that an identity between two numbers attributed to two different concepts "means the same" (i.e. "gleichbedeutend") as the equinumericity between the two concepts (i.e. the possibility of one-to-one correlation between the objects falling under these concepts). The short expression of it:

Number of F = Number of G means the same as F equinumerical with G

This is the first attempted definition of number in *Foundations of Arithmetic*, a book dedicated to the clarification of the concept of number and to the establishment of arithmetic on a purely logical foundation (the famous target of the logicist pro- gram). This definition will be rejected by Frege himself few paragraphs later and replaced with a second adopted definition for number in terms of extensions:

Let us try, therefore, the following type of definition; the direction of line a is the extension of the concept 'parallel to line a'; the shape of triangle t is the extension of the concept 'similar to triangle t'.

To apply this to our own case of Number, we must substitute for lines or triangles concepts and for parallelism or similarity the possibility of correlating one to one the objects which fall under the one concept with those which fall under the other. (Frege 1884, p. 76)

The result of the proposed replacement: "the number attributed to concept F is the extension of the concept 'equinumerical to the concept F'".

Given that extensions are thusly introduced as crucial to Frege's endeavour, this step from the first to the second definition- is particularly contentious and divisive. Rejected as it was by Frege, the first definition (i.e. Hume's Principle) has become a subject of debate and a nexus of various problems.

When considered as a possible alternative foundation for arithmetic (according to the Neo- Fregean proposal) its analytical character comes into question or its propriety as a definition. However, these issues will only be raised in passing in the current essay when they are related to the main objective, which is to find a Fregean defence for the use of extensions in the definition of numbers.

The part extensions play in Frege's system's inconsistency is well known and remains a topic of much debate. Most of the attempts of saving the Fregean enterprise or of continuing it, most of the solutions providing “a way out” of the contradiction are connected with the problem of extensions and with the utilization of the Axiom V from *Grundgesetze der Arithmetik.*

Defending the Role of Extensions: Ruffino and Demopoulos

In most cases, attempts to answer the question ‘why are extensions employed by Frege in his definition of number?’ are almost always accompanied by complaints about the sudden appearance of this solution or about its (allegedly) arbitrary character. Michael Beaney writes: “Having rejected the contextual attempt to provide an adequate definition of *direction* by means of (Da) and (Db), Frege’s alternative strategy is introduced without warning, and his proffered definition sprung on us with little explanation. In general, commentators point out Frege’s sudden choice, a decision that appears to have nothing to do with what has gone before. for example, states three main points regarding this problem: a) that given the sudden occurrence the definition using extensions seems an ad-hoc remedy (or, to quote her, “a cheat”); b) that nevertheless this usage is not so surprising because extensions play an important role in Frege’s work and are very frequently used among logicians of his time; c) that the strategy of using extensions “is mistaken” because it ultimately leads to paradox. Even if we assume that all of these assertions are true, it is still obvious that none of them can be used as an explanation.

The hypothesis that Frege was merely following a fashion among logicians of his time in employing extensions is, I believe, as an unlikely hypothesis, if we take into consideration at least two aspects. First, that it is somehow awkward to attribute Frege’s reasons for employing extensions to the fashion and habits of his time; this is mainly because he is famous for the revolution brought in the fashion and habits of seeing logic, mathematics and their instruments. However, it might be the case that even the most revolutionary authors have their points of continuity with the past and that one of these points could be the case of extensions.

Nevertheless, secondly, and more important, were extensions something ad-hoc adopted and not required by the inner structure of Frege’s system, then they should be also something easy to dismiss, when obvious that that they lead to difficulties. But this is obviously not the case, at least not for Frege (though the Neo-Fregean recent approach seems to suggest such a way out). In other words, if they were merely ornaments and not essential parts, their absence shouldn't render the Fregean alternative impossible. Frege, though, considered the failure to be serious.

One answer that is heard most often is that Frege uses extensions because they are seen as logical objects *sui generis* (if numbers are regarded as a certain kind of extensions, then this would be an important step forward for the logicist program). The subsequent step when agreeing upon this is to consider if there are reasons for maintaining this position about extensions. Wright tends to disagree, but one very elegant and convincing argumentation for the thesis of extensions as privileged logical objects can be found in Ruffino.

Ruffino starts from the observation that, even though there are not many indications in the Fregean text as to what is and what is not a 'logical object', there are indications that Frege considered concepts to be logical objects *par excellence*. Next, he notices that extensions of concepts and concepts were used by Frege, at least sometimes, as interchangeable items. That this substitution was not considered a problem by Frege at the time of *Foundations* (though later it becomes a quite considerable problem) is a tenet supported by the famous footnote from paragraph 68 in *Foundations*:

I believe that instead of 'extension of the concept' we could say simply 'concept'. But one would object to this in two ways:

1. This contradicts my previous statement that the individual number is an object, which is indicated by the definite article in expressions like 'the two' and by the impossibility of talking about ones, twos, etc. in the plural, as well as by the fact that the number can only be a part of a predicate in the statement of a number;

2. That concepts can have the same extension without coinciding I think, as it happens, that both objections can be met; but this would take us too far here. I assume that it is known what extension of a concept is. (Frege 1884, p. 80)

Ruffino explains the "interchangeability" between concepts and extensions by the fact that Frege uses extensions "as representative objects for concepts:

As the text suggests, in doing logic we have to speak of concepts, and in so doing, we have to 'transform' them into some special kind of objects, which are denoted by singular terms like 'the concept F'. These special objects are representative of concepts, and are introduced in logic as a necessary product of its discourse about concepts. (Obviously, 'transform' should be taken as a metaphor here, since, for Frege, concepts cannot be 'transformed' into objects or into anything else. What he means by this 'transformation' is the consideration of a concept not directly, but via its representative objects.) (Ruffino 2003, pp. 58–59) Or, Ruffino argues, concepts are essential tools of logic in a Fregean frame; their logicality accounts also for the logicality of extensions, not because they are identical but because, as Frege himself maintains, they are “very closely connected”:

(…) in Frege’s view, concepts are the main subject that logic has to deal with in unfolding the laws of truth. Concepts then have a special status as logical entities. (…) the paradigmatic nature of extensions as logical objects is derived, I submit, from the methodological and ontological primacy of concepts in logic. Their logicality comes from the fact that they are objects necessarily introduced by the activity of logical investigations. They are, in Frege’s view, a necessary product of doing logic, for doing logic necessarily involves talking about concepts and talking about concepts, requires, according to him, ‘trans- forming’ them into their representative objects. And these are extensions (…) (Ruffino 2003, p. 70)

Ruffino's conclusion is that, even if it is not technically essential, adding extension to the definition of number, of deducing the theorems of arithmetic in the system, was nevertheless essential for a broader, philosophical purpose, namely for the logicist project of reducing arithmetic to logic:
If extensions are the paradigmatic case of logical objects, and if the recognition of the logicality of an object is conditioned for Frege on its reducibility to extensions, then we can see why he could not have adopted the way out suggested by Wright and Boolos. Adopting Hume’s Principle as an axiom would be efficient from a technical point of view, but it would be incompatible with Frege’s concern to make evident the logical nature of arithmetic. A system composed of second-order logic plus Hume’s Principle would be consistent and would also yield all the relevant results of arithmetic, including the infin0ity of the series of natural numbers. But the recognition of arithmetic as a part of logic in this case would depend on blindly accepting numbers as logical objects, without any reduction to entities that were referred to in an essential way in logical theory as it is the case with extensions (or value-ranges). In Frege’s eyes, this would not be a convincing strategy at all, and would amount to giving up logicism. (Ruffino 2003, p. 71)

Demopoulos justifies the use of extensions on many reasons, but he also has a larger philosophical issue. His main concern is not so much the logicist project, but a unified theory of number, one including our technical and our ordinary (or, as he names it, “applied”) usage of numbers. In the previously presented account, the importance of the logicist program was easily transferable to the epistemic status of logic and arithmetic, regarding their analyticity or their a priori character: that was the upshot of the entire project. In Demopoulos’ case is not so much the a priori character, but the generality of the theory that comes under scrutiny (even though for a Kantian this would be hardly a surprise, because being a priori implies the properties of necessity and universality). His fundamental argument is that Hume's Principle is insufficient to provide a comprehensive explanation of number, an account that would also encompass the practical, everyday uses of number in addition to their theoretical applications:The difficulty is that although the pre analytic notion and its analysis in terms of Hume's principle are both compatible with making a stipulation which would exclude the identification of Julius Caesar with the number 4, pre analytically it is immediately evident that Julius Caesar is not a number. The fact that Hume' s principle is consistent with the possibility that Julius Caesar is a number therefore shows that it *fails to capture the whole of our pre analytic understanding.* (…)

I want to suggest that although the contextual definition fails to give a complete account of our notion of number, it does succeed in the case of our number-theoretic use of numerical singular terms, so that we should view Frege's Caesar problem as showing how clarity concerning our number-theoretic uses may fail to yield a full account of the matter, even when it succeeds in making evident the general fact that the numbers are applied in counting. *It will then emerge that the point of Frege's explicit definition was to fully capture that part of our use of numerical singular terms which arises in our applications of arithmetic; the introduction of the explicit definition therefore constitutes a genuine extension of Frege's theory of number*. None of this should be particularly surprising. Whatever novelty my account contains is cantered on the light it casts on the connection between the introduction of extensions and the Caesar problem, on the way in which it represents Frege's conception of the difference between a successful account of *all our uses* of numerical expressions (as opposed to an account which successfully captures *only the number-theoretic* case), and on the emphasis it places on the specific form of the definition in terms of extensions- the use of equivalence classes of concepts- rather than the general, categorical, fact that the definition is couched in terms of some notion of extension or other. (Demopoulos 1998, pp. 489–90)

Demopoulos asserts that using extensions allows us to specify inside theoretical applications of numbers and generalise across all uses of numbers, or would if it were to be successful. To explain: inside the technical frame of using numbers, i.e. regarding numbers only structurally, many accounts equally justified may be given and were given; adding extensions would enable the claim that one of these structures represents *the numbers*, namely the numbers as used also outside the technical frame (a claim that is, indeed, in line with the Fregean claim of objectivity of arithmetical notions). In this way, extensions also result in a generalisation, an effort to account for every application of numbers:

So long as the numbers are captured only 'structurally', there is a certain conventionalism which attaches to the assertion of their existence*: any ω-sequence* will serve as the sequence of numbers, but this is a feature without any analogue in our conception of the constituents of the physical world. A complete vindication of 'Fregean Platonism' would require being able to distinguish *the* numbers – a unique such sequence. But for this we require resources that go beyond the contextual definition; it is precisely at this point that the notion of extension is required (…).

The project of securing reference to the particular sequence of objects which are the natural numbers required the step to equivalence classes since it is unclear how, other than by some such device, one could fashion a definition that would 'comprehend' all applications of the numbers. Were *Grundlagen* expounding a 'pure' theory of number, rather than a theory which *aimed to cover both pure and applied statements of number, there would have been no need to introduce extensions.* (Demopoulos 1998, p. 492)

Although the justifications for supporting the usage of extensions vary, I think the authors mentioned above have several characteristics. As I shall attempt to demonstrate, these shared characteristics might lead to another broad philosophical reason that was not specifically stated and merely existed in the background.

Extensions, Numbers and the Universal Language of Thought:

The philosophical goal set forth by Demopoulos was the unification of technical usage and applied usage of numbers by identifying which of the possible alternative technical reconstructions of number is the one that is also present in the applied usage. The philosophical goal set forth by Ruffino was the unification of arithmetic and logic by discovering the logical structure of fundamental mathematical notions like "number" (i.e., the aim of the logicist programme).

Therefore, I think a further question can be asked: why this *impetus* towards unification of domains? Namely, if it is accepted that in order to fulfil the logicist dream, extensions are needed, a further question may appear: why aiming at reducing arithmetic to logic? And also, if we accept that extensions would give the reunification between the technical and the applied usage of numbers, why aiming at such a reunification? In this sense, an attempt of answering this further question is, I believe, a further justification of the role of extensions in the Fregean conceptual frame.

The opinion I want to argue for is that the impulse towards unification of different domains and the striving towards one and only one scheme of explanation comes from Frege's Leibnizian project of a *universal language of thought* conceived as an all-encompassing frame where all sciences have a certain place. From the perspective of a universal language of thought, extensions of concepts appear as a needed link providing the required unity of the domain, either between mathematics and logic or between different usages of the concept of number, usages that have to have the same explanatory frame, because it is one and only one true explanation describing the objective (i.e. intersubjective) reality of our thoughts.

In "Boole's logical Calculus and the concept-script" Frege draws several distinctions between his Begriffsschrift (concept-script) and Boole's enterprise. It is here where describes himself as inheriting and continuing Leibniz' grand project of a "universal language of thought":

In his writings, Leibniz threw out such a profusion of seeds of ideas that in this respect he is virtually in a class of his own. A number of these seeds were developed and brought to fruition within his own lifetime and with his collaboration, yet more were forgotten, then later rediscovered and developed further. (…)As part of this, I count an idea which Leibniz clung to throughout his life with the utmost tenacity, the idea of a *lingua characterica,* an idea which in his mind had the closest possible links with that of a *calculus ratiocinatur*. That it made it possible to perform a type of computation, it was precisely this fact that Leibniz saw as a principal advantage of a script which compounded a concept out of its constituents rather than a word out of its sounds, and of all hopes he cherished in this matter, we can even today share this one with complete confidence. (Frege 1880/81, p. 9) A few pages later he explains in more detail his position regarding the project and the relation that he sees as appropriate between *lingua caracterica* and *calculus ratiocinator*:

Right from the start I had in mind the expression of a content. What I am striving after is a lingua characterica in the first instance for mathematics, not a calculus restricted to pure logic.9 But the content is to be rendered more exactly than is done by verbal language. For that leaves a great deal to guesswork, even if only of the most elementary kind. There is only an imperfect correspondence between the way words are concatenated and the structure of the concepts. (Frege 1880/81, p. 12)

Usually, the written signs of natural languages are, first of all, signs for different sounds, and written words are accurate transcriptions of strings of sounds, which may have meanings and references attached, but these are obliterated. In other words, because the rules of combination for signs (written or spoken) are different from the rules of combination for ideas, the written natural language is not "trans- parent" for thoughts, its own structure being the one that obscures the structure of meanings expressed. Lingua characterica is a striving towards this "transparency", a striving to go beyond signs of language in order to "expose" thoughts. In a succinct characterization, "A lingua characterica ought, as Leibniz says, '*peindre non pas les paroles, mais les pensées'* (Frege 1880/81, p. 13). In both cases (i.e. Leibniz and Frege), though the results are different, the striving is the same: to bring the writ- ten sign as close as possible to thought so that the structure of a string of a signs would suggest the relations in a string of thoughts and in this way to make the newly designed language more accurate and suggestive than the old one; in the same time, the abstract, formal element and the identification of logical rules of combination would make computation possible and in this way a "calculus of ideas" would be possible. Obviously, the supposition here is that there is an objective, intersubjective level of thoughts that is supposed to be accurately described in only one manner by the level of signs. There is no place for conventionality here. One and only one description of our correct laws of thinking can be the true one.

This "language of thought" (conceived as both a language and a calculus, but not *primarily* a calculus) was supposed to be *accurate*, without gaps (due to the specification of rules of combination for signs) like a calculus and *universal* like a language. The opposition towards Boole's calculus comes, I believe, mainly from the lack of its universality, from its inability of being an all-encompassing frame for sciences, an instrument necessary for every discipline aspiring to *become science*. Frege's tenet about science was that "science is a system of truths"; the corollary is very simple: if we do not have a system, we do not have science. Therefore, *the role of logic seen as the universal language of thought becomes essential for the development of science*.

Noticeably, there are two positions that mathematics may have in respect to the universal language of thought: it can be a very part of it (because it would be reducible to logic) or it can be seen as a particular science that uses the instrument offered by logic (i.e. it has its own specific concepts incompatible with the maximum generality required by logic). If mathematics is thought in the first manner, then basic concepts, like the concept of number, must be definable in purely logical terms. For reasons well exposed by Ruffino above, extensions are good candidates for "purely logical terms".

However, this is not the only choice available. Frege started with the first claim (which is his main purpose in *Foundations*) but arrived towards the end of his life to the second, by saying that the sources of knowledge in mathematics are two: the geometric kind of knowledge and logic. I believe it is a point many times disregarded that the failure of the logicist program (if this is how we choose to interpret the contradiction in Frege's system) does not entail in any way the failure of the project of the universal language of thought. This is, today, an antiquated project. But for Frege at the time of *Foundations* and even later, it was the project underlying his entire work.

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