Review Article

Journal of Scientific Enquiry
Vol: 3 (1) , Pages: 60-65

Surface Covering with Regular Polygons: A Brief Discussion

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Received: 15.01.2023; accepted: 24.03.2023; published online: 30.06.2023

Abstract

In this article a limelight has been fallen on some of the possible ways to cover a surface (flat, curved and closed) by some of the regular polygons. To cover a two dimensional surface with regular polygons; equilateral triangle, square, and regular hexagon can be found to be used separately but regular pentagon, heptagon and other higher regular polygons cannot be used separately for this purpose. A curved surface can be found to be covered with regular polygons such as equilateral triangle, square, and regular pentagon separately but cannot be covered with hexagons or other higher regular polygons unless one or more polygons having size less than six are present. For covering a closed surface that is to make a curved surface closed it is found that equilateral triangle, square, and regular pentagon can be employed separately or various combination thereof. Regular polygons having size six or more in addition with regular polygons having size less than six can be used to cover a closed surface in a planned way governed by Euler's polyhedron formula.

Keywords: Regular polygon; Surface; Curved surface; Closed Surface; Polyhedra; Euler's polyhedron formula

1. Introduction:

Covering a two dimensional (flat) or three dimensional (open or closed) surface with some regular polygons becomes much more interesting as well as challenging in science, technology, arts as well as in our regular daily life. Covering a flat (2-dimensional) surface with some regular polygons is relatively easy whereas it becomes much more complicated in case of closed 3-dimensional surfaces [1, 2]. Euler is one of the great mathematicians who put forward a formula known as Euler's polyhedron formula [5] that gives how a closed three dimensional spherical surface can be covered by regular polygons. Before going in details let come across some important definitions.

2. Definitions

2.1. Vertex and edge

Vertex is commonly known as a point in a graph or in a diagram where edge(s) are incident. An edge is a line that connects two vertices. A vertex and an edge are shown in the following diagram given in Fig. 1 below.

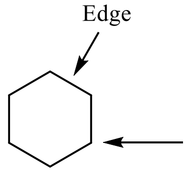

Fig. 1: Vertex and edge shown in regular hexagon

2.2. Polygon

'Poly' means 'many' and 'gon' means 'side'. So a polygon is an object enclosed by more than two sides or edges. Polygons are not allowed to have holes in them.

A polygon is called 'regular' if all of its sides have the same length, and all the angles between them are the same. Fig. 2 shows some examples of regular polygons.

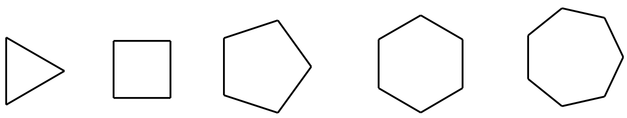

Fig. 2: Regular polygons

2.3. Surface

Surface is a portion of space that has length and breadth but no thickness. Surface may be two types: *orientable* and *nonorientable*. An *orientable* surface is one each sides have different surface normal vector whereas a *nonorientable* surface is one which has one side and one boundary and hence has one surface normal vector. An example of a *nonorientable* surface is Möbius strip [8-10]. A surface that contains ~~Möbius strip is by default is *nonorientable*. A~~ surface which is not *nonorientable* is obviously an *orientable* surface. In this article covering of an *orientable* surface is emphasized. An *orientable* surface can be of three types; flat, curved and closed.

3. Covering of a flat surface with regular polygons

*Email: skarmakarbu@rediffmail.com

 DOI: https://doi.org/10.54280/jse.233111

Let us start the covering of a flat surface with regular polygons of same kind. For such covering around a common vertex of embedded polygons must be equal to 2π . If it is greater than 2π the surface becomes convex and if it is less than 2π the surface becomes concave one. Keeping this in mind let us try to cover up a surface with regular polygons taking them in ascending order.

3.1. Covering with equilateral triangle

Equilateral triangles have internal angle of 60°. For six equilateral triangles to share a common vertex has the total angle around it $\varphi = 6 \times 60^0 = 360^0$. Thus the array is flat or planar as shown in Fig. 3.

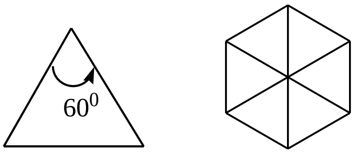

Fig. 3: Flat surface covering with equilateral triangles

3.2. Covering with squares

A square has internal angle of 90°, there is one possibility of placing four squares on a flat surface sharing a common vertex i.e. the four squares sharing a common vertex would lie in one plane, because total angle around a common vertex is $\varphi = 4 \times 90^0 = 360^0$ (as shown in Fig. 4).

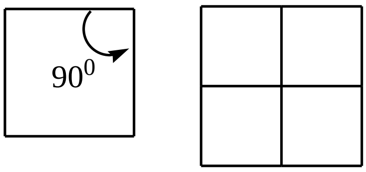

Fig. 4: Flat surface covering with squares

3.3. Covering with regular pentagons

With regular pentagons with internal angle of 108°, there is no possibility of covering a flat surface because three pentagons meeting at a common vertex results in angle (as shown in Fig. 5) $\varphi = 3 \times 108^0 = 324^0 \langle 360^0$. Placing of three pentagons sharing a common vertex will result a curved-surface rather than a flat surface. Four or more pentagons could not be fitted together as $\varphi = 4 \times 108^0 = 432^0 \rangle 360^0$. Such surface lifts from planarity and results in a curved surface.

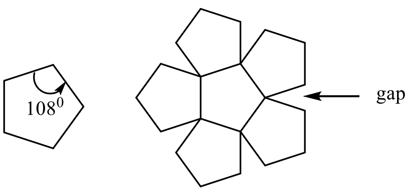

Fig. 5: Impossible to cover flat surface with regular pentagons

the angle (φ)

3.4. Covering with regular hexagons and with other higher polygons

With regular hexagons (internal angle: 120°), three hexagons sharing a common vertex lie in the same plane as in this case $\varphi = 3 \times 120^0 = 360^0$. Fig. 6 illustrates this situation.

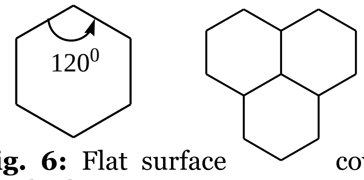

Fig. 6: Flat surface covering with regular hexagons

With heptagon and next higher regular polygons, there is also no chance to cover a flat surface as three of them cannot be fitted together at a common vertex.

4. Covering of a curved surface with regular polygons

If a surface is generated by stitching polygons in such a way that the total internal angle around a common vertex is less than 360° then the resulting surface is out of plane and hence produces a curved surface. Now let us discuss how the regular polygons separately or various of them can cover a curved surface.

4.1. Covering with equilateral triangles

Taking three equilateral triangles and stitching together in such away so that they meet at a vertex with total internal angle $180^0 (\langle 360^0)$ one can produce an out of plane surface (curved surface). Curved surface can also be obtained with taking 4 equilateral triangles meeting at a common vertex or 5 such triangles meeting at a common vertex. Six such triangles meeting at a common vertex results a plane surface with total internal angles equals to 360° around a vertex.

4.2. Covering with squares

When 3 squares meet at a vertex making total internal angle around the common vertex equals to 270° then their combination produces a curved surface. Again 4 squares meeting a common vertex gives a plane surface with angle equals to 360° around the common vertex.

4.3. Covering with regular pentagons

When three regular pentagons are stitched together so that they meet at a common vertex with total internal angles around the common vertex equals to $3 \times 108^0 = 324^0 (\langle 360^0)$ the resulting surface becomes a curved ones.

Three regular hexagons are fused together to result 360° internal angle around the common vertex so cannot be fused together to cover a curved surface. Regular polygons with size greater than six cannot result curved surface separately since three such polygons always result an internal angles more than 360° around the common vertex when they are fused together. But when such polygons having size six or more are stitched with one or more polygon(s) with size less than six, result will be covering up of curved surface.

5. Covering of a closed-surface with regular polygons

Closed surface is a kind of curved surface when the later gets closed. Closed surfaces with regular polygons are the examples of surfaces of 'regular or semi-regular polyhedrons or polyhedra'. 'Poly' is a Greek word means 'many' and 'hedron' is an Indo-European word means 'seat'. A regular polyhedron consists of only one type of regular polygon where as a semiregular one consists of more than one kind of polygons.

During the covering of a closed-surface by some regular polygons, the angle (φ) around a common vertex (as all vertices are equivalent) must be less than 2π.

Let us try of covering up of a regular polyhedron with regular polygons [2] and for this purpose let us choose the regular polygons ascending order in their size.

With equilateral triangles, there three possibilities: (a) three equilateral triangles with a common vertex ($\varphi = 3\times 60^\circ < 2\pi$) giving rise to tetrahedron, (b) four equilateral triangles with a common vertex ($\varphi = 4\times 60^\circ < 2\pi$) giving rise to octahedron and (c) five equilateral triangles with a common vertex ($\varphi = 5\times 60^\circ < 2\pi$) resulting in icosahedron.

With square there is only one possibility of stitching three squares with a common vertex $\varphi = 3\times 90^0 = 270^0 \langle 2\pi$ and this gives rise to cube.

With regular pentagon there is only one possibility of having three pentagons meeting at a common vertex as $\varphi = 3\times 108^0 = 324^0 \langle 360^0$ resulting in dodecahedron. Four or more pentagons could not be fitted together as $\varphi = 4\times 108^0 = 432^0 \rangle 360^0$.

With regular hexagons or with other higher polygons taking separately, there is no way to cover a curved surface as discussed in the earlier section and hence a closed-surface.

To cover a closed-surface of any semiregular polyhedron (Archimedean solid) by regular polygons of two or more mixed varieties, one of the following two points should be obeyed.

a) Combination of equilateral triangles, squares, regular pentagons may be used to cover up a semiregular polyhedron surface in a planned way such that the internal angle at each of the vertices in the surface is less than 360°.
b) Regular polygons with size six or more can be used to cover up such polyhedron if and only if a given number of regular polygons of size less than six is present required by the criterion of having internal angle at each vertex is of less than 360°.

To cover up a polyhedron (regular or semi-regular) surface, the number of polygon(s) and their variety can well be understood with the use of Euler's Polyhedron formula [5]. Euler mentioned his result in a letter to Goldbach (Goldbach's Conjecture Problem) in 1750. However he did not give the correct proof of his formula. Now-a-days there are several proof of Euler's Polyhedron formula. Graph theory [11] may also be employed to achieve such formula. B. Mandal derived the formula using the concept of graphical tree [4]. The above conclusions regarding covering up of closed surfaces are found to be the direct consequence of this well-known formula. The formula is given as,

$$F - E + V = 2$$

where F is number of faces, E is the number of edges and V is the numbers of vertices of a polyhedron. These numbers of faces, edges and vertices are required to cover up the said polyhedron.

5.1. Regular polyhedral or Platonic solids

5.1.1. Covering a polyhedron with equilateral triangle

Let us assume that number of regular triangular faces required to cover up a polyhedron surface is x and degree of each vertex is d. Then the number of vertices (V) and edge (E) are:

$V = 3x/d$ (since each face has three vertices and again d number of faces share a common vertex).

$E = 3x/2$ (since each face has three edges and again each edge is being shared by two such faces).

Introducing all these values of faces, edges and vertices in Euler's polyhedron formula we have,

$$x - \frac{3x}{2} + \frac{3x}{d} - 2 = 0 \Rightarrow x = \frac{4d}{6-d} \quad ....(1)$$

Now if we put $d = 3$ in eqn (1) i.e. 3 equilateral triangles sharing a common vertex then $x = 4$ i.e. 4 equilateral triangles will be required to cover the closed-surface giving rise to a tetrahedron (Fig. 7).

If we put $d = 4$ in eqn (1) i.e. 4 equilateral triangles sharing a common vertex then $x = 8$ i.e. 8 equilateral triangles will be required to cover the closed-surface giving rise to an octahedron (Fig. 7).

If we put $d = 5$ in eqn (1) i.e. 5 equilateral triangles sharing a common vertex then $x = 20$ i.e. 20 equilateral triangles will be required to cover the closed-surface giving rise to a icosahedron (Fig. 7).

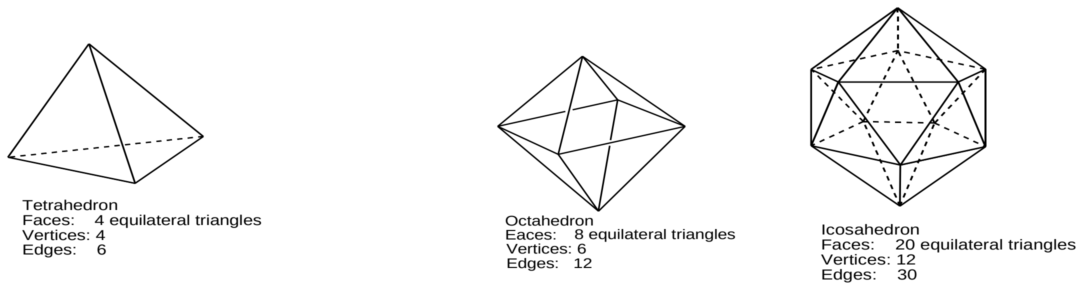

Fig. 7: Covering tetrahedron, octahedron and icosahedron with equilateral triangles

5.1.2. Covering a polyhedron with square

Let $F = x =$ Number of squares required to cover the closed-surface. Then

$E = \frac{4x}{2}$ (the factor $\frac{1}{2}$ arises as each edge is being shared by two squares)

$V = \frac{4x}{3}$ (the factor $\frac{1}{3}$ arises as each vertex is being shared by three squares).

Now according to Euler's polyhedron formula we have,

$x - \frac{4x}{2} + \frac{4x}{3} - 2 = 0 \Rightarrow x = 6$, i.e. 6 squares will be required to cover the closed-surface giving rise to a cube (Fig. 8).

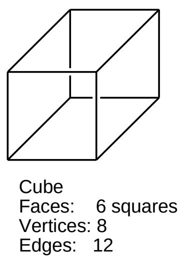

Fig. 8: Covering a cube with squares

5.1.3. Covering a polyhedron with regular pentagon

Let us try to cover a closed-surface with regular pentagons and assume that the number of such pentagon is $F = x$ then

$E = \frac{5x}{2}$ (the factor $\frac{1}{2}$ arises as each edge is being shared by two regular pentagons)

$V = \frac{5x}{3}$ (the factor $\frac{1}{3}$ arises as each vertex is being shared by three regular pentagons).

Thus in this case the Euler's polyhedron formula reduces to

$x - \frac{5x}{2} + \frac{5x}{3} - 2 = 0 \Rightarrow x = 12$, i.e. 12 regular pentagons will be required to cover the closed-surface giving rise to a dodecahedron (Fig. 9).

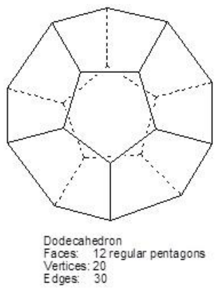

Fig. 9: Covering a dodecahedron with regular pentagons

5.2. Semiregular Polyhedra or Archimedean solids

5.2.1. Covering up with equilateral triangles and squares simultaneously

Let us try to cover up a closed-surface (polyhedron) with some equilateral triangles and squares simultaneously with total number of faces, $F = x = t + s$, where t and s be the required number of equilateral triangles and squares respectively. Then,

$E = \frac{3t + 4s}{2}$ (since each edge is shared by two polygons)

$V = \frac{3t + 4s}{4}$ (since each vertex is shared by four polygons)

Substituting all these in Euler's polyhedron formula we have

$$(t + s) - \left(\frac{3t + 4s}{2}\right) + \left(\frac{3t + 4s}{4}\right) - 2 = 0 \Rightarrow t = 8$$

....................(2)

As s cancels out on simplification of eqn.(2) so it may have any integral value: 0, 1, 2, 3, ... We thus get a very important result that in order to cover a closed-surface with some equilateral triangles and squares, number of equilateral triangles required is exactly 8 and the number of squares is arbitrary.

Thus, with $t=8$ and $s=0$ (i.e. number of squares is zero), we get an octahedron.

5.2.2. Covering up with regular pentagons and hexagons simultaneously

If we try to cover up a closed-surface (polyhedron) with some regular pentagons and some regular hexagons simultaneously with total number of faces, $F=x=p+h$ where p and h be the required number of pentagons and hexagons respectively.

$E=\dfrac{5p+6h}{2}$ (since each edge is shared by two polygons)

$V=\dfrac{5p+6h}{3}$ (since each vertex is shared by three polygons)

Substituting all these in Euler's polyhedron formula we have

$$(p+h)-\left(\frac{5p+6h}{2}\right)+\left(\frac{5p+6h}{3}\right)-2=0\Rightarrow p=12 \quad ....(3)$$

As h cancels out on simplification of eqn.(3) so it may have any integral value: 0, 1, 2, 3, ... We thus get a very important result that in order to cover up a polyhedron with some regular pentagonal and hexagonal faces, number of regular pentagons required is exactly 12 and the number of hexagons is arbitrary.

Thus, with $p=12$ and $h=0$ (i.e. number of hexagons is zero), we get a dodecahedron. Here it is important to note that if all the vertices of this polyhedron are replaced by C-atoms, the polyhedron becomes the smallest fullerene, C_{20} [3].

If we put $p=12$ and $h=20$, we get a structure, called truncated icosahedron. That is such polyhedron can be obtained by truncation of 12 vertices (of degree 5) of icosahedron resulting 12 pentagonal faces that are separated to each other by 20 hexagonal faces. The formation of truncated icosahedron from a icosahedron is shown in Fig.10. If all the vertices of the truncated icosahedron are replaced by C-atoms it becomes [60] Fullerene (C_{60}) named after renowned architect Buckminster Fuller [3]. Fullerene was isolated from soot of graphite laser vapouration in an inert atmosphere [6,7]. It is very important material for both theoretical and experimental investigations.

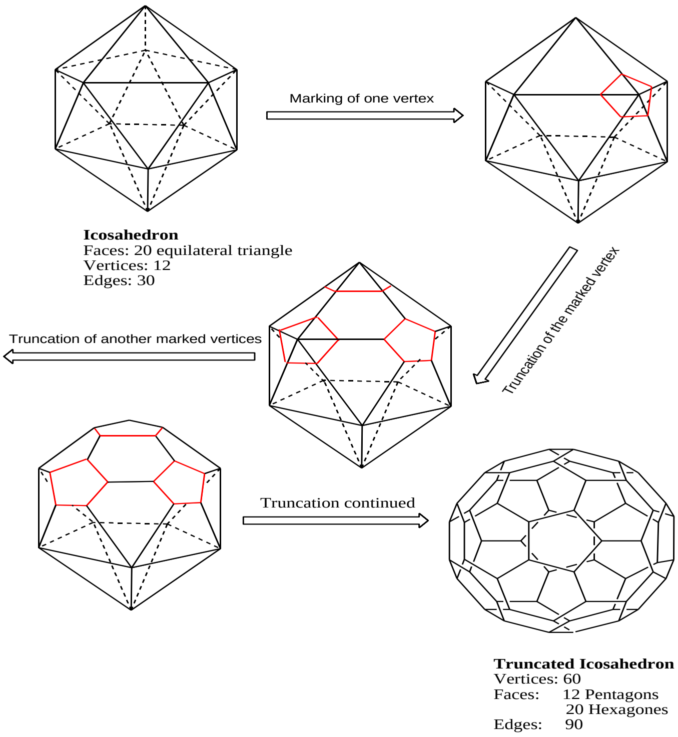

Fig. 10: Formation of truncated icosahedron from a icosahedron

5.2.3. Covering up with regular heptagons and regular pentagons simultaneously

If we try to cover up a closed-surface with some regular pentagons and some regular heptagons simultaneously with total number of faces, $F=x=p+h$ where p and h be the required number of pentagons and heptagons respectively, then

$E=\dfrac{5p+7h}{2}$ (since each edge is shared by two polygons)

$V=\dfrac{5p+7h}{3}$ (since each vertex is shared by three polygons)

Substituting all these in Euler's polyhedron formula we have

$$\left(p+h\right)-\left(\frac{5p+7h}{2}\right)+\left(\frac{5p+7h}{3}\right)-2=0 \Rightarrow p-h=12$$

................(4)

For example fourteen numbers of regular pentagons and two heptagons produce a closed surface on fusion as shown in Fig. 11

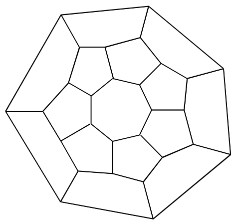

Fig. 11: Planar representation of a closed surface made up of fourteen regular pentagons and two heptagons

Conclusion:

Thus, it may be concluded that

(i) A flat surface can only be covered with equilateral triangles, squares, regular hexagons separately but not with pentagons and other higher polygons having size greater than six separately.

(ii) A curved surface can be covered only with equilateral triangles, squares, regular pentagons separately or any combination thereof but not with hexagons and higher regular polygons separately. Hexagons and other higher regular polygons can be used to cover curved surface if and only if there is one or more regular polygon(s) having size less than six.

(iii) A closed surface (polyhedron surface) can be covered with equilateral triangles, squares, pentagons and various combination thereof in a planned way governed by Euler' polyhedron formula. Hexagons and other higher polygons can be used to cover such surface but in that case certain numbers of polygon of size less than six are needed as required by the Euler's polyhedron formula.

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